COMPRESSIONS IN ABELIAN AND NON-ABELIAN GROUPS

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I am very thankful to my supervisor for his encouraging, inspiring and loving guidance throughout this work in his hard pressed health conditions.

C O M P R E S S I O N S

I N

A B E L I A N AND N O N - A B E L I A N G R O U P S

PREFACE

This dissertation entitled 'Compressions In Abelian And Non-Abelian Groups' is the research work done by me since 23rd of August, 1963 in the Department of Mathematics and Statistics, A.M.U.Aligarh.(India) under the encouraging, inspiring and close supervision of Prof. M.A.Kazim, Reader in the Department. The work was all along financially supported by the council of scientific and indusrial research, Govt. of India.

The concepts of normality and conjugacy which play a fundamental role in the structural study of non-abelian groups are incidently completely trivial in the case of abelian groups. This, infact, led me to think seriously to fill up the vast gap in that direction. In this attempt, the concept of 'Compression' is introduced and studied in several of its various aspects (though not completely for lack of space) in this humble presentation. The power of the concept both in abelian and non-abelian groups will be to a larger extent evident through the pages of this work. The technique used is not always very sophisticated but mostly very simple and easily readable which was sometimes found essential for the growth of the concept. A number of other concepts based on this study are introduced like self compression, complete self compression (§5, chapter II), c.s.c-subgroups, smallest c.s.c-subgroups (chapter III), essential equality (Def.3.3), C-simple groups (Def.3.4), compressor (Def.4.1), c-power of an element (Def.4.3) compression classes, locally c.s.c-groups (Def.5.1), power C-transform (Def.5.2), generalised compressor (Def.6.1); compression series (Def.7.3), compression chains (Def.7.5), C-measure and invert C-measure (Def.9.4, 9.3) and some others in the sequel.

At the outset I must acknowledge my deep indebtedness and respect to my supervisor for his continued influence, and my due homage to Prof. R. Baer and Prof. B.H.Neumann who have thrilled me on occasions by their works which I have used, and the deep thanks for the encouragement and suggestions I received from them in my personal contacts through correspondence. I should also thank Prof. G.Higman, with whom I came in close contact when preparing the work during three months seminar at Chandigarh (Punjab, India.), for his constructive criticism which enlightened me a lot.

The thesis consists of two parts : Part I - Abelian groups Part II - Non-abelian groups and comprises ten chapters with one appendix out of which first seven chapters are devoted to the study of compression in abelian groups on the non-abelian group pattern which has facilitated the direct use of arguments from abelian groups to non-abelian groups. All those theorems proved in these chapters while found holding true even in non-abelian groups have been classified at the end of each chapter. The last three chapters exclusively deal with compression in non-abelian groups. The results proved here are in addition to those already remarked to be valid both for abelian and non-abelian groups.

In all nearly 180 Theorems accompanied by nearly 80 corollaries and several remarks have been established.in the course of this dissertation. To satisfy the requirements of clause VIII of chapter XXV Academic ordinances, every chapter is equiped with a comprehensive introduction in the begining pointing out the main theorems proved and a bibliography at the end of it indicating the sources used, with the contents articlewise classified and tabulated in the Table of Contents.

In chapter I, we introduce the concept of 'Compression' and study the basic properties and other behaviours of C-transforms of subsets and subgroups. In theorem 1.14, we characterise images of a pair of subsets under a homomorphism as a pair of compression subsets. Chapter II deals with self compression a special form of the concept of compression. We develope the concept in several directions and give a criterion for self compression of all non-trivial subgroups of a periodic group. Chapter III studies the properties of the smallest completely self compressed (c.s.c-) subgroup and of groups which have no proper c.s.c-subgroups(C-simple groups). Since the smallest c.s.c-subgroup is the basic one as far as the concept of compression goes, it has been studied in details in its several aspects. The theorems on relations between the rank of a group and of its smallest c.sc-subgroup, the index of the smallest c.s.c-subgroup in the group are of great interest (theorem 3.6 to 3.8) and are used in later study. The notions of essential equality and C-simplicity are quite interesting in their applications. In chapter IV a detailed study of compressor, the analogue of normaliser, obtained from the concept of compression has been made in details. A quite interesting theory of the ascending chain of compressors is developed with the help of the concept of c-power of an element (§11). Chapter V provides a glance into compression classes and some results parallel to those proved in an interesting but simple paper of B.H.Neumann (Bibliography - Chapter V, (10)). A new concept of power-C-transform (Def.5.2) is introduced to study compression classes by a finer tool. In Chapter VI a slight generalisation of the concept of compressor is studied and in Chapter VII we deal with compression series and chains, and obtain some interesting results.

The content of Chapters VIII, IX & X is the continued study of the problems of compression in non-abelian groups similar to those of Abelian groups. Some interesting study is made pertaining to equivalence of this relation of compression in non-abelian cases. Moreover, some results parallel to (Bibliography- Chapter VIII (7) and (8)) are proved in chapter VIII. Several criteria for self compression of subsets and subgroups are provided in chapter IX with various other intrinsic properties ..e.g. the compressor is the normal subset of the normaliser. An interesting study of compressor is made in chpater X and some really interesting situations have been dealt regarding the relations of compressor and normaliser.

Lastly the material of this dissertation has not yet been sent for publication as the author finds still a lot of scope to work in different directions of its aspects in abelian and non-abelian groups utilising the theory of compression developed here. However some work of the author closely related to this work has been either published or is to be published in near future. These papers are on Translative mappings((1) and (2) appendix)and on a generalisation to normality((3), appindex) is to appear in. the Mathematics Student.

It will be most appropriate to thank the C.S.I.R., Government of India, for the financial help given to me for this work. I would also like to thank my friends and colleagues Dr. F. Husain, Mr.Md. Nasiruddin, Mr. Afzal Ahmad Khan, Mr. Amit Roy, Mr. Nirenjan Singh, Mr. Ved Prakash Misra, Mr. K.C.Mittal and Dr. Rangaswamy, with whom I had occasions to exchange ideas and who helped me at different stages. My thanks are also due to Prof. Siddiqi, Chairman of Department of Mathematics,for the facilities he provided and the pains he took for the completion of this work. I appreciate the courtesy and help of the commerce Department A.M.U., and duely thank to Mr. Mukhtar N.Khan and Mr. Ved Prakash who helped me in typing the manuscript.

To conclude, I am again in extreme obligations and express my deep sense of gratitude to my supervisor for his valuable attention inspite of his preoccupations, his sympathetic criticism and loving care throughout his guidance.

P K Sharma

(P.K. SHARMA)

A.M.U. Aligarh.

June ,1966.

TABLE OF CONTENTS

PART - ONE

(Abelian Groups)

PART - TWO

(Non-Abelian Groups)

(The copies of 2 and 3 will be supplied as soon as the papers are published)

LIST OF NOTATIONS

$S, T, X, \ldots\ldots, G, H, \ldots\ldots$	:	Groups (or their Subsets)
$a, b, c, g, h, s, t, x, y, \ldots\ldots$	:	Elements of Groups.
I	:	Ring of Integers.
$\{a, b, \ldots\}$	:	Set of elements $a, b, \ldots\ldots$
$[a, b, \ldots]$	:	Subgroup generated by $a, b, \ldots\ldots$
$O(a)$	:	Order of a
$\|S\|$ or $O(S)$	:	Power of S.
$r(G)$	:	Rank of G.
$r_o(G)$	:	Torsion free rank of G.
$r_p(G)$	:	p-rank of G.
$A \subseteq B$ $(A \subset B)$	:	A is a (Proper) subset of B.
$A \cup B$, $A \cap B$	:	Union, Intersection of sets A & B.
$[G : H]$	:	Index of subgroup H in G.
$G \cong H$	:	G and H are isomorphic.
$G \simeq H$	:	H is a homomorphic image of G.
$\underset{i}{X}\, G_i$	:	Direct product (external or internal) of groups.
$\prod_{\alpha \in A} S_\alpha$	:	Set product of subsets.
$a \xrightarrow{c} b$	:	b is a compression of a
G^*	:	The smallest C.S.C. - subgroup of G.
$AC(G)$	:	Anticenter of the group G.

$R(G)$	:	Rim of G.
S_x	:	Compressed transform of S w.r. to x.
$Z(G)$	:	Centreof G.
$Z(S)$	:	Centralizer of thesubset S.
$S^{(g)}$	:	Power C-transform of S w.r. to g.
C.S.C.-	:	Completely self compressed.
$\Leftrightarrow$ or iff	:	if and only if
$\Rightarrow$	:	implies.
$C_G(S)$	:	Compressor of the subset S in G.
$N_G(S)$	:	Normaliser of S in G.
S/K	:	Quotient of the set S by the set K.
$G-S$	:	The complement of S in G.
$\forall$	:	For every.
Ker. ϕ	:	Kernel of ϕ
(i, j)	:	g.c.d. of i and j.
$G[n]$	:	Set of all g in G s.t. $ng = 0$
G_p	:	p-component of G.
$\overset{-}{\underset{-}{\leftarrow - \rightarrow}}$	:	Essential equality.
G/H	:	Factor group of G by H.
$C_E(g, S)$	:	Compression exponent of g w.r. to S.

$C^{S_1}_{S_2}$	:	S_1- compressor - S_2 or Generalised compressor of (S_1, S_2).
$[a,b]^*$	:	Invert C-measure.
$\overline{[a,b]}$	:	C-measure.
G'	:	Derived subgroup of G.
ϕ	:	Empty set or a map
$S(G)$	:	Socle of G

Part I
(Abelian Groups)

CHAPTER - ONE

COMPRESSED - TRANSFORMS

1. Introduction: We know that in a group G, an element $a \in G$ is called conjugate to an element $b \in G$ if there exists an element $x \in G$ such that $x^{-1}ax = b$. This relation has no importance in abelian groups, since in an abelian group an element is conjugate to itself only. We, introduce a new concept of ' compression ' analogous to that of conjugacy and thereby are in a position to study even abelian groups on nearly the same lines as the non-abelian groups are studied through the relation of conjugacy. The strength of the concept of 'compression' and the role it plays in the study of both abelian and non-abelian groups will be exhibited and exmined in different chapters of this dissertation. We begin this chapter with the definition of 'compression' for elements in a group and generalise it to sets which we shall call 'compressed transforms'. We find that the relation of 'compression' is an equivalence relation in an abelian group. In this chapter we study, in details, the calculus and other important properties of compressed transforms such as : the condition of their being subgroup, the nature of their collections, their order with respect to original set, their equality, their behaviour under fundamental mappings and also in direct products etc; and bring out that the order of a set and that of its compressed transform is the same; the compressed transform of a subgroup need not be a subgroup whereas a conjugate transform of a subgroup is a subgroup;

however, if the compressed transform of a subgroup happens to be a subgroup, it must coincide with the subgroup itself (Theorem 1.8). We also prove that the compressed transform of a semigroup with identity in a group form a group with respect to set product and this group is a homomorphic image of the given group, and further find a criterion for a pair of subsets to be a pair of compression subsets under a homomorphism in the image group.

In the sequel, we shall always regard G, throughout part I, a multiplicative abelian group and shall denote its identity by e unless otherwise mentioned.

2. Concept of Compression:

Def. 1.1 - Let $a, b \in G$, then if there exists an element $x \in G$, such that

$$xax = b$$

we say that b is a 'compression' of a with respect to x, in symbols $a \xrightarrow{C} b$ implying that 'a is compressed to b'.

In general we define :

Def. 1.2 - Let S and K be any two subsets of a group G. The subset K will be called " compression " or " compressed transform " or " C -transform " of the subset S by an element $x \in G$ if

$$xSx = K \qquad (S \xrightarrow{C} K)$$

We shall denote the compressed transform of S by x as S_x.

We can easily verify that compressions of any two distinct subsets in a group G with respect to a fixed element of G are always distinct; moreover every subset of G, in general, cannot be put as a compression of a given subset of G. For example:

If $G = [a]$; the infinite cyclic group generated by a.

Then

$$a \overset{C}{\not\longrightarrow} a^2 \quad \text{i.e.} \quad a^2 \text{ is not a compression of } a.$$

3. Compression As An Equivalence Relation:

Theorem. 1.1 - The relation of ' compression ' is an equivalence relation in a group G.

Proof. Let S, K, T be any subsets of G.

(i) Reflexivity : $S \xrightarrow{C} S$.

$$\text{For } e \text{ in } G, \; S = eSe$$

(ii) Symmetry : $S \xrightarrow{C} K \Longrightarrow K \xrightarrow{C} S$

There exists an element $x \in G$ such that

$$xSx = K$$

$$\Longrightarrow S = x^{-1} K x^{-1}$$

$$\Longrightarrow K \xrightarrow{C} S$$

(iii) Transitivity: If $S \xrightarrow{C} K$ and $K \xrightarrow{C} T \Longrightarrow S \xrightarrow{C} T$

There exist elements $x, y \in G$ such that

$$K = xSx \quad \text{and} \quad T = yKy$$

$$\Longrightarrow T = y(xSx)y$$
$$= xy.\ S.\ xy \quad (\text{ G is abelian})$$
$$\Longrightarrow S \xrightarrow{C} T$$

This completes the proof of theorem.

Remark: The relation of compression is also an equivalence relation between elements of a group G, hence we can partition the group G into equivalence classes with respect to this relation. We shall study these classes, in details, in a later chapter.

4. C - Transforms And Their Calculus:

In what follows, we shall investigate the behaviour of C-transforms of union, intersection, product, complimentation and quotient of subsets of a group G.

Theorem 1.2 - If $(S_\alpha)_{\alpha \in A}$ be any family of subsets of a group G where A is an index set then, for any $x \in G$, we have

$$\text{(i)} \quad (\bigcap_{\alpha \in A} S_\alpha)_x = \bigcap_{\alpha \in A} (S_\alpha)_x \quad \text{(ii)} \quad (\bigcup_{\alpha \in A} S_\alpha)_x = \bigcup_{\alpha \in A} (S_\alpha)_x$$

Proof. (i) Firstly it is clear, that

$$(\bigcap_{\alpha \in A} S_\alpha)_x \subseteq (S_\alpha)_x \text{ for every } \alpha \in A$$

$$\Longrightarrow (\bigcap_{\alpha \in A} S_\alpha)_x \subseteq \bigcap_{\alpha \in A} (S_\alpha)_x$$

On the otherhand, if $s \in \bigcap_{\alpha \in A} (S_\alpha)_x$ then,

$$s = xs_\alpha x \quad \text{where} \quad s_\alpha \in S_\alpha \,, \quad \alpha \in A$$

Thus, for any $\alpha_i, \alpha_j \in A$, there exist $s'_{\alpha_i} \in S'_{\alpha_i}$, $s'_{\alpha_j} \in S'_{\alpha_j}$ such that

$$s = xs'_{\alpha_i}x = xs'_{\alpha_j}x$$

$$\Longrightarrow s'_{\alpha_i} = s'_{\alpha_j}$$

Hence,

$$s \in (\bigcap_{\alpha \in A} S_\alpha)_x$$

$$\Longrightarrow \bigcap_{\alpha \in A}(S_\alpha)_x \subseteq (\bigcap_{\alpha \in A} S_\alpha)_x$$

Consequently,

$$(\bigcap_{\alpha \in A} S_\alpha)_x = \bigcap_{\alpha \in A}(S'_\alpha)_x .$$

(ii) Since,

$$S_\alpha \subseteq \bigcup_{\alpha \in A} S'_\alpha \quad \text{for every } \alpha \in A$$

$$\Longrightarrow (S_\alpha)_x \subseteq (\bigcup_{\alpha \in A} S'_\alpha)_x$$

$$\Longrightarrow \bigcup_{\alpha \in A} (S_\alpha)_x \subseteq (\bigcup_{\alpha \in A} S'_\alpha)_x$$

Next if,

$$s \in (\bigcup_{\alpha \in A} S'_\alpha)_x$$

There exists an element $s_\alpha \in \bigcup_{\alpha \in A} S_\alpha$ such that

$$s = x s_\alpha x \text{ where } s_\alpha \in S_\alpha \text{ for some } \alpha \in A$$

$$\Longrightarrow s \in \bigcup_{\alpha \in A} (S_\alpha)_x$$

$$\Longrightarrow (\bigcup_{\alpha \in A} S_\alpha)_x \subseteq \bigcup_{\alpha \in A} (S_\alpha)_x$$

Hence,

$$(\bigcup_{\alpha \in A} S_\alpha)_x = \bigcup_{\alpha \in A} (S_\alpha)_x$$

This completes the proof.

Cor. 1.1 - If $(S_\alpha)_{\alpha \in A}$ be any family of subsets of a group G such that $\bigcap_{\alpha \in A} S_\alpha = \phi$, then $\bigcap_{\alpha \in A} (S_\alpha)_x = \phi$ for any $x \in G$.

Theorem 1.3 - If $(S_\alpha)_{\alpha \in A}$ be a finite family of subsets of a group G and $(x_\alpha)_{\alpha \in A}$ be a set of elements in G, then the set product.

$$\prod_{\alpha \in A} (S_\alpha)_{x_\alpha} = (\prod_{\alpha \in A} S_\alpha)_{\prod_{\alpha \in A} x_\alpha}$$

Proof. Let $s_\alpha \in S_\alpha$ be arbitrary then since G is abelian,

$$\prod_{\alpha \in A} (x_\alpha s_\alpha x_\alpha) = \prod_{\alpha \in A} x_\alpha \cdot \prod_{\alpha \in A} s_\alpha \cdot \prod_{\alpha \in A} x_\alpha \in (\prod_{\alpha \in A} S_\alpha)_{\prod_{\alpha \in A} x_\alpha}$$

$$\Longrightarrow \prod_{\alpha \in A} (S_\alpha)_{x_\alpha} \subseteq (\prod_{\alpha \in A} S_\alpha)_{\prod_{\alpha \in A} x_\alpha}$$

On the contrary, since every element of $\prod_{\alpha \in A} S_\alpha$ is of the form $\prod_{\alpha \in A} s_\alpha$ and,

$$\prod_{\alpha \in A} x_\alpha \cdot \prod_{\alpha \in A} s_\alpha \cdot \prod_{\alpha \in A} x_\alpha = \prod_{\alpha \in A} (x_\alpha s_\alpha x_\alpha)$$

$$\Longrightarrow (\prod_{\alpha \in A} S_\alpha)_{\prod_{\alpha \in A} x_\alpha} \subseteq \prod_{\alpha \in A} (S_\alpha)_{x_\alpha}$$

Hence

$$\prod_{\alpha \in A} (S_\alpha)_{x_\alpha} = (\prod_{\alpha \in A} S_\alpha)_{\prod_{\alpha \in A} x_\alpha}$$

This completes the theorem.

Cor 1.2 - If $(S_\alpha)_{\alpha \in A}$ be a finite family of subsets of a group G then for any $x \in G$,

$$(\prod_{\alpha \in A} S_\alpha)_x = (S_{\alpha'})_x \prod_{\alpha (\neq \alpha') \in A} S_\alpha = \prod_{\alpha (\neq \alpha') \in A} S_\alpha \; (S_{\alpha'})_x$$

Cor. 1.3 - Let S, K, T be any subsets of a group G then for any $x \in G$

(i) $(S_x \cup K_x)\, T = (S \cup K)\, T_x$

(ii) $(S_x \cap K_x)\, T = (S \cap K) T_x$

(The result clearly follows if we consider C-transforms of $(S \cup K) T$ and $(S \cap K) T$).

Theorem 1.4 - Let S be any subset of a group G and $x \in G$, then

$$(G-S)_x \subseteq G - S_x \text{ where } G\text{-}S \text{ is the complement}$$

of S in G.

Proof. We know by Cor. 1.1, that

$$(G-S)_x \cap S_x = \phi \quad \text{since } (G-S) \cap S = \phi$$

This implies

$$(G-S)_x \subseteq G - S_x$$

Hence the proof follows.

Def. 1.3 - We shall call (S/K) to be the quotient of the set S by the set K in a group G if whenever $x \in (S/K)$, $xK \subseteq S$.

Theorem 1.5 - If S and K be any two subsets of a group G and $x \in G$ then

$$(S/K) = (S_x/K_x)$$

Proof. Firstly if $y \in (S/K)$

$$\implies yK \subseteq S$$
$$\implies x(yK)x \subseteq xSx$$
$$\implies y(xkx) \in S_x \quad \forall\, k \in K$$
$$\implies y \in (S_x/K_x)$$
$$\implies (S/K) \subseteq (S_x/K_x)$$

Again if

$$y' \in (S_x/K_x)$$
$$\implies y'(xkx) \in S_x \quad \forall\, k \in K$$

$$\Longrightarrow \quad y'k \in S$$

$$\Longrightarrow \quad y' \in (S/K)$$

$$\Longrightarrow \quad (S_x/K_x) \subseteq (S/K)$$

Hence the result follows.

Cor. 1.4 - If S, K be any two subsets of a group G and $x \in G$, then

$$(S/K)K_x = (S_x/K_x)_x\ K \subseteq S_x$$

(It is evident since $(S/K)\ K \subseteq S$)

Cor.1.5 - For any three subsets S,K,T of a group G and $x \in G$,

$$((S/K) \cup T)_x = (S_x/K_x)_x \cup T_x$$

$$((S/K) \cap T)_x = (S_x/K_x)_x \cap T_x .$$

5. Power of A Set And that of Its C-Transform :

The following theorem establishes the equality of the cardinality of a set and of any of its C-transforms. Thus it simplifies the study of C-transforms particularly in finite groups.

Theorem 1.6 - A subset S if a group G and its C-transform S_x with respect to any $x \in G$ have the same cardinal number.

Proof. We have

$$S_x = x.S.x$$

Now consider the mapping

$$s \longrightarrow xsx \quad \text{where } s \in S .$$

Evidently this mapping is single-valued and onto. Also if for $s_1, s_2 \in S$

$$xs_1x = xs_2x$$

$$\implies s_1 = s_2$$

Thus the mapping is also 1-1

$\implies$ S and S_x have the same cardinal numbers.

Cor. 1.6 - If G be a finite group then for any subset S of G and $x \in G$

$$S_x \supseteq S \text{ or } S \supseteq S_x \text{ implies } S_x = S.$$

In view of the above theorem, theorem 1.4 is modified as follows :

' If G is finite, $G - S_x = (G-S)_x$ '

This is immediate since cardinality of $G-S_x$ and $(G-S)_x$ is the same.

6. C-Transform - A Subgroup:

We first establish a criterion for a C-transform of a subset to be a subgroup and then point out the circumstances in which a C-transform of a subgroup be again a subgroup.

Theorem 1.7 - Let S be a subset of a group G then a C-transform S_x of the set S with respect to $x \in G$ is a subgroup if and only if $s_1, s_2 \in S$ implies $s_1 s_2^{-1} \in S_x$.

Proof. Firstly let S_x be a subgroup then if $s_1, s_2 \in S$,

$$(xs_1x)(xs_2x)^{-1} = (xs_1x)(x^{-1}s_2^{-1}x^{-1})$$

$$= xs_1s_2^{-1}x^{-1}$$

$$= s_1s_2^{-1} \quad \text{since G is abelian}$$

$$\Longrightarrow s_1 s_2^{-1} \in S_x$$

Conversely, if the condition be satisfied, then clearly arguing backward S_x is a subgroup of G.

This establishes the theorem.

Cor. 1.7 - Let S be a subset of G and S_x be a subgroup of G with $S \supseteq S_x$, then S is a subgroup of G.

Theorem 1.8 - Let H_1, H_2 be any two subgroups of a group G and $x \in G$ then if $(H_1)_x = H_2$ implies $H_1 = H_2$.

Proof. Since $(H_1)_x = H_2$

$$\Longrightarrow x^{-2} \in H_2$$

Also

$$(H_1)_x = H_2$$

$$\Longrightarrow H_1 = (H_2)_{x^{-1}}$$

$$= H_2 . x^{-2}$$

$$= H_2$$

Hence the proof is complete.

7. Equality of A Set And of It's C-Transforms:

We can easily observe from Theorem 1.6, that for every finite subsets S of a group G and any $x \in G$ if $S_x \supseteq S$ or $S \supseteq S_x$ then $S_x = S$, but this phenomenon is not true, in general, in case of infinite subsets. For example, if we pick up two infinite subsets S_1 and S_2 in an infinite group we find that $(S_1)_x \subset S_1$ and $(S_2)_x \supset S_2$.

Example. Let $G = [a]$, the infinite cyclic group generated by a. $S_1 = \{a^2, a^4, a^6, \ldots\}$, $S_2 = \{a^{-2}, a^{-4}, a^{-6}, \ldots\}$ be two infinite subsets and $x = a$. Then,

$$(S_1)_x = \{a^4, a^6, a^8, \ldots\} \subset S_1$$

$$(S_2)_x = \{e, a^{-2}, a^{-4}, \ldots\} \supset S_2$$

We further find that the equality $S_x = S$ holds always in case S be any subgroup but in case of infinite subsets, the equality is available if $O(x) < \infty$. It is interesting to note that if $O(x) = \infty$, we observe the presence of subsets S_1, S_2 in relation to the infinite subset S such that $S_1 \subseteq S$, $S_2 \supseteq S$ for which $(S_1)_x = S_1$ and $(S_2)_x = S_2$. Lastly we also find a criterion for equality of C-transforms of a given subgroup with respect to different elements in G.

Theorem 1.9 - Let H be any subgroup of a group G and $x \in G$, then

$$H_x \supseteq H \quad \text{or} \quad H \supseteq H_x \text{ implies } H_x = H$$

Proof. Firstly let,

$$H_x \subseteq H$$

$$\Longrightarrow x^{-1} h x^{-1} \in H \quad \forall h \in H$$

$$\Longrightarrow h \in xHx$$

$$\Longrightarrow H \subseteq H_x$$

Hence

$$H_x = H$$

On the otherhand if,

$$H \subseteq H_x$$
$$\Longrightarrow H_{x^{-1}} \subseteq H$$

Hence, as above

$$H_{x^{-1}} = H$$
$$\Longrightarrow H = H_x$$

This proves the theorem.

Theorem 1.10 - Let S be any subset of a group G and $x \in G$, then

(i) If $O(x) < \infty$, $S_x \subseteq S$ or $S \subseteq S_x$ implies $S_x = S$.

(ii) If $O(x) = \infty$, $S_x \subseteq S$ or $S \subseteq S_x$ implies there exists a subset S_1 of S such that $(S_1)_x = S_1$ and also a subset S_2 of G containing S such that $(S_2)_x = S_2$.

Proof. Case (i) Let $O(x) = n$ and $S_x \subseteq S$, then

$$S_{x^i} \subseteq S \;\forall \text{ +ve integers } i \text{ and } S_{x^i} \downarrow S \text{ as } i \uparrow n$$

$$\Longrightarrow S \subseteq S_x \subseteq S$$

$$\Longrightarrow S_x = S$$

On the contrary if

$$S \subseteq S_x$$

$$\Longrightarrow S_{x^{-1}} \subseteq S$$

Hence, since $O(x) = O(x^{-1}) < \infty$, we have as above

$$\Longrightarrow S_{x^{-1}} = S$$

$$\Longrightarrow S = S_x$$

This proves (i)

Case (ii) Let $O(x) = \infty$ and $S_x \subseteq S$, then

$S_{x^i} \subseteq S \;\forall\; i \in I,\ i > 0$ and S_{x^i} is a

decreasing sequences as $i \uparrow \infty$

$$\Longrightarrow \bigcap_{i=1}^{\infty} S_{x^i} \subseteq S$$

We put

$$S_1 = \bigcap_{i=1}^{\infty} S_{x^i}$$

$$\Longrightarrow (S_1)_x = \bigcap_{i=1}^{\infty} S_{x^{i+1}} \quad \text{by theorem 1.2 (i)}$$

$$= \bigcap_{i=1}^{\infty} S_{x^i} \quad \text{since } S_{x^i} \downarrow$$

$$= S_1$$

Again if

$$S_x \subseteq S$$

$$\Longrightarrow S \subseteq S_x^{-1}$$

$$\Longrightarrow S \subsetneq S_x^{j} \quad \text{where } j \in I,\ j < 0 \text{ and } S_x^{j} \uparrow \text{as } j \downarrow -\infty$$

Let $S_2 = \bigcup_{i=1}^{\infty} S_x^{-i}$

$$\Longrightarrow (S_2)_x = \bigcup_{i=1}^{\infty} S_x^{-i+1} \quad \text{by theorem 1.2 (ii)}$$

$$\Longrightarrow (S_2)_x = \bigcup_{i=1}^{\infty} S_x^{-i} \quad \text{since } S_x^{-i} \uparrow \text{ and } S_x^{-1} \supseteq S$$

$$= S_2$$

Finally, if $S \subseteq S_x$ the proof follows arguing as in case (i).

Thus the proof is complete.

Theorem 1.11 - Let H be a subgroup of a group G then for any $x,y \in G$, $H_x = H_y$ if and only if $(xy^{-1})^2 \in H$.

Proof. Firstly let

$$H_x = H_y$$

then there exists $h, h' \in H$ such that

$$xh'x = yhy$$

$$\Longrightarrow x^2h' = y^2h$$

$$\Longrightarrow (xy^{-1})^2 = hh'^{-1} \in H$$

Conversely if the condition be satisfied then,

$$(xy^{-1})^2H = H$$

$$\Longrightarrow \quad x^2 H = y^2 H$$

$$\Longrightarrow \quad H_x = H_y$$

This completes the theorem.

Remark. The above theorem can also be stated as follows:

'Let H be a subgroup of a group G, then for any $x,y \in G$, $H_x = H_y$ if and only if to every $h \in H$ there exists $h' \in H$ such that $(xy^{-1})^2 = hh'^{-1}$'.

We can easily verify that for an arbitrary subset S of a group G the above condition is necessary but not sufficient in general. To check that it is not sufficient in general, consider

$G = [a]$, The infinite cyclic group generated by a and $S = \{a^{-2}, a^{-4}, a^{-6}, \ldots\}$ an infinite subset. Then if we take

$x = a^{-2}$, $y = a^{-1}$, we find that to every $s \in S$, $\exists$ $s' \in S$ such that

$$(xy^{-1})^2 = ss'^{-1}$$

But, however, we have $S_x \subset S_y$

If S be a finite subset then, in view of theorem 1.6, the condition is also sufficient.

8. C-Transforms And Fundamental Mappings:

In this section, we prove that the collection of all C-transforms of a semigroup S with identity in a group G is a group with respect to set multiplication and is a homomorphic image of G. We also prove that a homomorphic image of a C-transform of a subset with respect to an element in G is a C-transform of the homomorphic image of the subset with respect to the image element. Finally, we deduce a necessary and sufficient condition for a pair of subsets to be a pair of compression subsets under a homomorphism.

Theorem 1.12 - Let S be a semigroup with identity in a group G, then the set $K = \{ S_x \mid x \in G \}$ is a group with respect to set multiplication, and this group is a homomorphic image of G, the kernel of homomorphism being the set of all elements $x \in G$ such that $S_x = S$.

Proof. (i) Closure : Let S_x, $S_y \in K$ then,

$$S_x . S_y = (S.S)_{xy} \text{ by theorem 1.3}$$
$$= S_{xy} \in K$$

($S^2 = S$, since S is a semigroup with identity)

(ii) Associativity : The set product in a group is always associative.

(iii) Identity : Let S_x be any C-transform in K, then since $S = S_e \in K$, we have

$$S_x . S_e = S_e . S_x$$

$$= (S.S)_x$$

$$= S_x$$

$\Longrightarrow$ S is identity in K

(iv) Inverse: $S_x . S_{x^{-1}} = S_{x^{-1}} . S_x$

$$= (S.S)_{x^{-1}x}$$

$$= S_e$$

$$\Longrightarrow S_{x^{-1}} = (S_x)^{-1}$$

This proves that K is a group.

We now define a mapping ϕ of G to K as :

$$\phi : x \longrightarrow S_x$$

Evidently ϕ is single valued and onto. It is a homomorphism, since for $x_1, x_2 \in G$

$$(x_1 x_2)\phi = S_{x_1 x_2}$$

$$= (S.S)_{x_1 x_2}$$

$$= S_{x_1} . S_{x_2} \text{ by theorem 1.3}$$

$$= (x_1)\phi\ (x_2)\phi$$

The kernel of the homomorphism is the set of all x for which

$$(x)\phi = S_e = S$$

i.e. $\mathrm{Ker}\ \phi^* = \{ x \in G \mid S_x = S \}$

This proves the theorem completely.

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* We have defined this set as $C_G(S)$, see chapter - four

Cor.1.8 - Let H be a subgroup of a group G then the set

$$K = \{ H_x \mid x \in G \} \text{ is a subgroup of } G/H.$$

(This follows at once since $H_x = x^2 H \in G/H$)

Theorem 1.13 - Let ϕ be a homomorphism of a group G onto a group G' and S be a subset of G, then if $x \in G$, we have

$$(S_x)\phi = (S\phi)_{x\phi}$$

Proof. Let $s \in S$ be arbitrary, then

$$(x\phi)(s\phi)(x\phi) = (xsx)\phi \in (S_x)\phi$$

$$\Longrightarrow \quad (S\phi)_{x\phi} \subseteq (S_x)\phi$$

Also,

$$(xsx)\phi = (x\phi)(s\phi)(x\phi) \in (S\phi)_{x\phi}$$

$$\Longrightarrow \quad (S_x)\phi \subseteq (S\phi)_{x\phi}$$

Hence

$$(S_x)\phi = (S\phi)_{x\phi}$$

The assertion is proved.

If S_1, S_2 are any two subsets of a group G, the question arises ' Whether homomorphic images of these subsets can be pair of compression subsets? If so, under what conditions? This is answered in the following theorem:

Theorem 1.14 - Let ϕ be a homomorphism of a group G onto a group G' and S_1, S_2 be any two subsets of G then $(S_1)\phi \xrightarrow{C} (S_2)\phi$

if and only if for some $x \in G$, $(S_1)_x \subseteq S_2K$ and $(S_2)_{x^{-1}} \subseteq S_1K$ where K is Ker. ϕ'.

Proof. Firstly if,

$$(S_1)\phi \xrightarrow{C} (S_2)\phi$$

Then $\exists$ an element $x' \in G'$ such that

$$x'.(S_1)\phi.x' = (S_2)\phi$$

$$\Longrightarrow \quad (x\phi).(S_1)\phi.(x\phi) = (S_2)\phi \text{ where } x \in G \text{ s.t. } x\phi = x'$$

$$\Longrightarrow \quad (xS_1x)\phi = (S_2)\phi$$

$$\Longrightarrow \quad (S_1)_x \subseteq S_2K$$

Also,

$$(x\phi)(S_1)\phi(x\phi) = (S_2)\phi$$

$$\Longrightarrow \quad (S_1)\phi = (x^{-1}\phi)(S_2\phi)(x^{-1}\phi)$$

$$\Longrightarrow \quad (S_1)\phi = (x^{-1}S_2x^{-1})\phi$$

$$\Longrightarrow \quad (S_2)_{x^{-1}} \subseteq S_1K$$

Conversely, the proof is trivial, since

$$(S_1)_x \subseteq S_2K$$

$$\Longrightarrow \quad (S_1\phi)_{(x\phi)} \subseteq (S_2)\phi \quad \text{by theorem 1.13}$$

And

$$(S_2)_{x^{-1}} \subseteq S_1K$$

$$\Longrightarrow \quad (S_2\phi)_{(x\phi)^{-1}} \subseteq (S_1)\phi \quad \text{by theorem 1.13}$$

$$\Longrightarrow \quad (S_2)\phi \subseteq (S_1\phi)_{x\phi}$$

Consequently, under $x\phi \in G'$

$$(S_1)\phi \xrightarrow{C} (S_2)\phi$$

This completes the proof.

Cor.1.9 - Let ϕ be an isomorphism of a group G onto a group G' and S_1, S_2 be any two subsets of G then $S_1 \xrightarrow{C} S_2$ iff $(S_1)\phi \xrightarrow{C} (S_2)\phi$. (It follows immediately since Ker. ϕ is identity)

9. Direct Product And C-Transforms:

We consider in this section relation between C-transforms of subsets in different groups with C-transform of the resultant subset in the external direct product of those groups.

Theorem 1.15 - Let G_i, $i = 1,2, \ldots n$ be n groups and S_i be a subset of G_i for every i then for $x_i \in G_i$ for every i, we have

$$\underset{i}{X}(S_i)_{x_i} = (\underset{i}{X}\, S_i)_{(x_i)} \text{ where } (x_i) = (x_1, x_2, \ldots, x_n)$$

Proof. Let $s_i \in S_i$, $i = 1,2, \ldots n$ be arbitrary then for $(s_i) = (s_1, s_2, \ldots s_n)$ and $(x_i) = (x_1, x_2, \ldots, x_n)$ consider,

$$(x_i)(s_i)(x_i) = (x_i s_i x_i) \in \underset{i}{X}(S_i)_{x_i}$$

$$\Longrightarrow (\underset{i}{X}\, S_i)_{(x_i)} \subseteq \underset{i}{X}(S_i)_{x_i}$$

On the other hand

$$(x_i s_i x_i) = (x_i)(s_i)(x_i) \in (\underset{i}{X}\, S_i)_{(x_i)}$$

$$\Longrightarrow \quad \underset{i}{X}(S_i)_{x_i} \subseteq (\underset{i}{X}\, S_i)_{(x_i)}$$

Hence,

$$\underset{i}{X}(S_i)_{x_i} = (\underset{i}{X}\, S_i)_{(x_i)}$$

This completes the proof.

The above theorem is proved only for finite n but it can be generalised similarily for complete direct product of any family of groups.

10. Remark on Non-Abelian Groups:

As stated earlier, we have proved the results for abelian groups only, but it can be easily verified that the following theorems of this chapter :

1.2, 1.4, 1.6, 1.9, 1.10, 1.13, 1.14 and 1.15

also hold for non-abelian groups without any substantial change in the proofs already supplied.

CHAPTER - TWO

SELF COMPRESSED SUBSETS AND SUBGROUPS

1. Introduction : Having dealt with compressed transforms in the previous chapter, we now introduce the idea of 'self compressed' subsets and subgroups'. We call a subset S of a group G 'self compressed' with respect to an element $x \in G$ if its C-transform $S_x = S$, and generalise this idea to 'self compression' with respect to arbitrary subsets of G (Def.2.1). We investigate several criteria for an element, a subset or a subgroup of a group to be 'self compressed' with respect to an element or a subset (Theorems 2.1, 2.2, 2.3, 2.4). Further, we study other properties and aspects of self compressed subsets e.g. their calculus, their behaviour under fundamental mappings, their role in relation to direct products and the nature of their collections etc. Some of the important investigations are that ' an element of a group is self compressed with respect to all the elements of the group iff every element of the group is self compressed with respect to all the elements in the group ', also the collection of all self compressed subsets with respect to a given subset form a semigroup under set multiplication, and that such semigroups corresponding to any two different elements of a group are the same iff the cyclic groups generated by their squares are the same. We also determine a necessary and sufficient condition for self compression of all subgroups ($\neq e$) of a periodic group and prove a theorem

analogous to Zassenhaus lemma in case of self compressed subgroups which we shall use later in our study.

2. Notion Of Self Compression And Immediate Observations:

Def.2.1 - (i) An element a of a group G is called 'self compressed' with respect to an element $x \in G$,* if $xax = a$. In general, if there exists a subset K of G such that $yay = a$ for all $y \in K$, then a is called self compressed with respect to K, in particular if $K = G$, a will be called 'completly self compressed element' (c.s.c-element) in G.

(ii) In like manner a subset S of a group G is called 'self compressed' with respect to an element $x \in G$, if

$$xSx = S \quad (S_x = S)$$

If, however, there exists a subset K of G such that $S_y = S$ for all $y \in K$, then S is called self compressed with respect to K, in particular if $K = G$, S will be known as <u>completly self compressed subset</u> (c.s.c - subset) of G. We can also define, similarily, <u>self compressed</u> and <u>completly self compressed</u> subgroups, if we take S to be a subgroup H of G.

The following observations are not only interesting but also useful and can be easily verified.

(i) Every subset of G is self compressed with respect to identity element in G and also with respect to every element of order 2 in G.

(ii) The set of all self compressed elements with respect to any given element $x \in G$ is either the whole group G, or the null set. The former situation will arise only in case $O(x) = 2$.

(iii) Every element $x(\neq e) \in G$, which is self compressed with respect to itself, must necessarily be of order 2.

(iv) If for a subgroup H of G and for any $x \in G$, $H_x = H$, then the cosets of H with respect to x and x^{-1} are the same.

(v) An element of G is completly self compressed iff every element of G is of order 2.

(vi) A subset S of G is self compressed with respect to $x \in G$ if and only if it is self compressed with respect to x^{-1}.

(vii) Any coset of a subgroup H of G is completly self compressed, iff every coset of H is completly self compressed.

3- Characterizations For Self Compression:

Firstly, we find out necessary and sufficient conditions for self compression of subsets and subgroups of a group, and point out that ' any subgroup of a group is self compressed with respect to an element of the group iff every element of the subgroup is a compression of some element of the subgroup with respect to the element of the group. In case of subsets, the property is true only for finite subsets' . Secondly, we prove that a subgroup generated by the squares of the elements of a subset is self compressed with respect to that subset, and generally, any subgroup is self compressed with respect to a subset iff it contains the subgroup generated by the squares of the elements of the subset. Finally, we provide a criterion for self compression of all <u>non-trivial</u> subgroups of a periodic group with respect to a given subset. We now prove the following theorems:

Theorem 2.1 - A finite subset S of a group G is self compressed with respect to an element $x \in G$, iff for every $s_1 \in S$ there exists an element $s_2 \in S$ such that $s_1 = xs_2x$.

Proof. Firstly, if S be self compressed with respect to $x \in G$,

$$S_x = xSx = S$$

$\implies$ For any $s_1 \in S \ \exists \ s_2 \in S$ such that

$$s_1 = xs_2x$$

Conversely, if $s_1 \in S$ be arbitrary, $\exists \ s_2 \in S$ such that

$$s_1 = xs_2x$$

$$\implies \quad S \subseteq xSx = S_x$$

$$\implies \quad S = S_x \quad \text{by theorem 1.6}$$

This completes the proof.

Remark : The above theorem does not hold, in general, for infinite subsets. For example

Let $G = [a]$, The infinite cyclic group generated by a.

Let $S = \{a, a^3, a^5, a^7, \ldots.\}$

and take $x = a^{-1}$

Then,

$$S_x = \{a^{-1}, a, a^3, a^5, \ldots..\}$$

Here, though the condition of the above theorem is satisfied, yet $S_x \supset S$.

The situation, however, is controlled if the set be an arbitrary subgroup.

Cor.2.1 - Any subgroup H of a group G is self compressed with respect to an element $x \in G$ if and only if for every $h_1 \in H$ there exists an element $h_2 \in H$ such that $h_1 = x\,h_2 x$.

(The proof follows in view of theorem 1.9)

The following theorem provides a necessary and sufficient condition for self compression of an arbitrary subset of a group with respect to an element.

Theorem 2.2 - Let S be a subset of a group G, then $S_x = S$ for any $x \in G$ if and only if $x^i\, S\, x^i = S$ for every $i \in I$ where I is the set of all integers.

Proof. Firstly, if

$$x^i\, S x^i = S \quad \text{for every} \quad i \in I$$

Then evidently for $i = 1$,

$$S_x = S$$

Again, if $S_x = S$, then recursively

$$S_{x^i} = S \text{ for all } i \in I,\ i \geq 0.$$

This is also true for $i < 0$, since

$$S_x = S \Longrightarrow S = S_{x^{-1}}$$

$$\Longrightarrow S = S_{x^{-j}} \text{ for all } j \in I,\ j > 0$$

This proves the result.

Theorem 2.3 - Let S be a subset of a group G, then $[S^2]$, the subgroup generated by squares of elements of S, is self compressed with respect to S; and any subgroup H of G is self compressed with respect to S if and only if $H \supseteq [S^2]$.

Proof. Let $s \in S$ be arbitrary, then clearly,

$$s[S^2]s = s^2[S^2] = [S^2]$$

$\Longrightarrow [S^2]$ is self compressed with respect to S.

For the other part, if H be any subgroup of G self compressed with respect to S, then

$$sHs = H \quad \forall\ s \in S$$

$$\Longrightarrow \quad ses = s^2 \in H$$

$$\Longrightarrow [S^2] \subseteq H$$

Conversely, if $H \supseteq [S^2]$, then for every $s \in S$

$$sHs = s^2H$$
$$= H$$

$\Longrightarrow$ H is self compressed with respect to S.

This proves the theorem completly.

Cor.2.2 - Let S_1, S_2 be any two subsets of a group G such that $[S_1^2] \subseteq [S_2^2]$, then if a subgroup H of G is self compressed with respect to S_2, it is self compressed with respect to S_1.

Remark. The theorem actually asserts that $[S^2]$ is the smallest self compressed subgroup with respect to the subset S.

Cor. 2.3 - A subgroup H of a group G is self compressed with respect to a subset S iff $s^2 \in H$ for all $s \in S$.

Theorem 2.4 - Every subgroup $H\ (\neq e)$ of a periodic group G is self compressed with respect to a subset K of G iff $yxy = x^i$, $i \in I$ for every $y \in K$ and any non-identity $x \in G$.

Proof. If every subgroup $H\ (\neq e)$ of G is self compressed with respect to K, then if $[x]$ be the cyclic subgroup generated by $x(\neq e) \in G$,

$$y[x]y = [x] \quad \forall\, y \in K$$

$$\Longrightarrow yxy \in [x]$$

$$\Longrightarrow yxy = x^i \quad \text{for some } i \in I$$

Conversely, suppose the condition be satisfied. Let H be any subgroup of G different from identity. For $h(\neq e) \in H$, we have

$$[h] \subseteq H$$

$$\Longrightarrow \quad yhy = h^i \in H \quad \forall\, y \in K$$

Further let $O(h) = n$,

$$\begin{aligned} yey &= yh^n y \\ &= yhy.\, h^{n-1} \\ &= h^{i+n-1} \in [h] \subseteq H \end{aligned}$$

Thus $yhy \in H$ for every $h \in H$

$$\Longrightarrow \quad yHy \subseteq H$$

$$\Longrightarrow \quad H_y = H \qquad \text{(Theorem 1.9)}$$

Hence H is self compressed with respect to K.

This was to be proved.

4. Calculus Of Self Compressed Subsets And Subgroups:

We shall discuss in this section, the behaviour of union, intersection and complementation of self compressed subsets including their generated subgroups.

Theorem 2.5 - Let $(S_\alpha)_{\alpha \in A}$ be any family of self compressed subsets of a group G with respect to a subset K of G then so are the subsets $\bigcap\limits_{\alpha \in A} S_\alpha$ and $\bigcup\limits_{\alpha \in A} S_\alpha$.

Proof. Let $y \in K$ be arbitrary, then applying theorem 1.2,

$$(\bigcup_{\alpha \in A} S_\alpha)_y = \bigcup_{\alpha \in A}(S_\alpha)_y = \bigcup_{\alpha \in A} S_\alpha$$

and

$$(\bigcap_{\alpha \in A} S_\alpha)_y = \bigcap_{\alpha \in A}(S_\alpha)_y = \bigcap_{\alpha \in A} S_\alpha$$

Thus $\bigcup_{\alpha \in A} S_\alpha$ and $\bigcap_{\alpha \in A} S_\alpha$ are self compressed with respect to K.

Cor. 2.4 - The intersection of any family of self compressed subgroups of a group G with respect to a subset K is self compressed subgroup with respect to K.

If, in particular, K = G, the above theorem holds true for c.s.c-subsets or subgroups.

Theorem 2.6 - Let S be a self compressed subset of a group G with respect to a subset K of G, then so is G-S, the compliment of S in G.

Proof. Let $y \in K$ be arbitrary, we have, by theorem 1.4,

$$(G-S)_y \subseteq G - S$$

Also, since

$$S_y = S \Longrightarrow S = S_{y^{-1}}$$

$$(G-S)_{y^{-1}} \subseteq G - S \qquad \text{(Theorem 1.4)}$$

Hence

$$(G-S)_y = G-S \quad \forall\, y \in K$$

This proves the result.

Theorem 2.7 - Let S be a subset of a group G, then for any $x \in G$, $S_x = S$ implies $([S])_x = [S]$, but not conversely.

Proof. Let $S = (s_\alpha)_{\alpha\in A}$ and $s_{\alpha_1}^{\epsilon_1}\, s_{\alpha_2}^{\epsilon_2} \ldots s_{\alpha_k}^{\epsilon_k}$ where

$\epsilon_i \in I$, $\alpha_i \in A$ for all $i = 1,2, \ldots k$ be any element of $[S]$.

Now, since $S = S_x$, we have $x\, s_{\alpha_1}\, x \in S$, $\alpha_1 \in A$, hence

$$x(s_{\alpha_1}^{\epsilon_1} \cdot s_{\alpha_2}^{\epsilon_2} \cdots s_{\alpha_k}^{\epsilon_k})x = (xs_{\alpha_1}x)\, s_{\alpha_1}^{\epsilon_1 - 1}\, s_{\alpha_2}^{\epsilon_2} \ldots s_{\alpha_k}^{\epsilon_k} \in [S]$$

$$\Longrightarrow x.[S].x \subseteq [S]$$

$$\text{i.e. } ([S])_x \subseteq [S]$$

$$\Longrightarrow [S] = ([S])_x \quad \text{(Theorem 1.9)}$$

Finally, to prove that the converse is not true,

Consider $G = [a]$, the infinite cyclic group generated by a.

Here evidently

$$a[a]a = [a] \quad \text{but } a.aa \neq a$$

Thus the falsety of the converse statement is established.

5. Criteria For Complete Self Compression :

The concept of complete self compression is an important concept. We investigate various Criteria in this direction and find out that if any element of a group is completly self compressed, then the group is generated only by elements of order 2 and conversely. Further, we characterize a subgroup as completly self compressed, if it contains a c.s.c-subset, and also establish that it is necessary and sufficient for a subgroup to be completly self compressed that its factor group be elementary abelian with elements of order 2 only. Finally, we give a criterion for complete compression of all non-trivial subgroups of a periodic group.

Theorem 2.8 - An element of a group G is completly self compressed iff G is generated by elements of order 2.

Proof. Let an element $a \in G$ be completly self compressed, then for every $x \in G$,

$$xax = a$$

$$\Longrightarrow x^2 = e$$

Hence G is generated by elements of order 2.

Conversely, if G is generated by elements of order 2, then every element of G is of order 2, since G is abelian therefore for any $b \in G$ and every $x \in G$

$$xbx = x^2b$$
$$= b$$

Thus every element of G is completly self compressed.

Cor.2.5 - Every subset of a group G is completly self compressed iff G is generated by elements of order 2.

Cor. 2.6- Any element of a group G is completly self compressed iff every element of G is completly self compressed.

(The proof is implied in the proof of the theorem itself)

Theorem 2.9 - A subgroup H of a group G is completly self compressed if and only if H contains a c.s.c-subset S of G.

Proof. Firstly, if H be a c.s.c-subgroup, it trivially follows that H contains a c.s.c-subset of G. Conversely, if $S \subseteq H$

$$\Longrightarrow [S] \subseteq H$$

Further, since S is a c.s.c-subset, [S] is c.s.c-subgroup in G by theorem 2.7. Hence, by Cor.2.3

$$x^2 \in [S] \subseteq H \quad \forall x \in G$$

$$\Longrightarrow H \text{ is c.s.c-subgroup}$$

This proves the theorem.

Theorem 2.10 - A subgroup H of a group G is completly self compressed if and only if G/H is elementary abelian with elements of order 2.

Proof. If G/H is elementary abelian with elements of order 2, then for any coset $xH \in G/H$, we have

$$(xH)^2 = H$$
$$\implies x^2 \in H$$

Hence by Cor.2.3, H is completly self compressed in G. Conversely, if H be c.s.c-subgroup in G then, we have, for any $x \in G$.

$$(xH)^2 = H$$
$$\implies xH \text{ is of order } 2.$$

$\implies$ G/H is elementary abelian with elements of order 2.

This proves the theorem.

Theorem 2.11 - Every subgroup $H(\neq e)$ of a periodic group G is completly self compressed iff $\bar{x}yx = y^{-1}$ for every $x, y(\neq e) \in G$.

(Proof follows immediately from theorem 2.4)

6. Totality Of Self Compressed Subsets:

We observe that the family of all self compressed subsets with respect to a given subset in a group form a semigroup with respect to set product, and further obtain a necessary and sufficient condition for equality of the semigroups corresponding to two different elements of the group.

Theorem 2.12 - The set $\overline{K}$ of all self compressed subsets of a group G with respect to a subset K of G is a semigroup with set multiplication as operation.

Proof. If S_1, $S_2 \in K$, then for any $y \in K$,

$$(S_1 S_2)_y = (S_1)_y S_2$$

$$= S_1 S_2 \text{ since } S_1 \in K$$

$$\Longrightarrow S_1 S_2 \in K$$

Also, since set multiplication is associative, K is a semigroup.

This proves the theorem.

Theorem 2.13 - Let G be a group and $x, y \in G$ then $\Sigma_x = \Sigma_y$ iff $[x^2] = [y^2]$ where Σ_x and Σ_y are the semigroups of all self compressed subsets of G with respect to x and y respectively.

Proof. Let $[x^2] = [y^2]$

$$\Longrightarrow y^2 = x^{2j} \quad \text{for some } j \in I$$

Now if $S \in \Sigma_x$,

$$\Longrightarrow S_x = S$$

$$\Longrightarrow S_{x^i} = S \quad \text{for all } i \in I \text{ (Theorem 2.2)}$$

$$\Longrightarrow S\,x^{2j} = S$$

$$\Longrightarrow S_y = S$$

$$\Longrightarrow s \in \Sigma_y$$

$$\Longrightarrow \Sigma_x \subseteq \Sigma_y$$

Similarly, we get

$$\Sigma_y \subseteq \Sigma_x$$

$$\Longrightarrow \Sigma_x = \Sigma_y$$

Conversely, let $\Sigma_x = \Sigma_y$

$$\Longrightarrow y[x^2]y = [x^2]$$

$$\Longrightarrow y^2 \in [x^2]$$

$$\Longrightarrow [y^2] \subseteq [x^2]$$

On the other hand, we can similarily verify that

$$[x^2] \subseteq [y^2]$$

$$\Longrightarrow [x^2] = [y^2]$$

This completes the proof.

Cor.2.7 - Let G be a group, then for any $x \in G$, $\Sigma_x = \Sigma_{x^i}$ for every $i \in I$ provided $(i, O(x)) = 1$.

Proof. (Proof trivially follows since $(i, O(x)) = 1$ implies $[x^2] = [x^{2i}]$ if $O(x) < \infty$).

7. Fundamental Mappings On Self Compressed Subsets:

In this section, we first establish that the homomorphic image of a self compressed subset of a group with respect to another subset of the group is self compressed with respect to image subset. The converse of this theorem holds always in case the homomorphism is an isomorphism. We, further, show that if a group G' be the homomorphic image of a group G, then there exists one to one correspondence between the self compressed subgroups of G containing the kernel with respect to a given subset and the self compressed subgroups of G' with respect to image subset-a result already known for normal subgroups. We note, however, that this property is only one way true for arbitrary subsets of the group. The other important result that we point out is that there exists a homomorphism of all subsets with identity of a group onto the collection of smallest self compressed subgroups with respect to these subsets.

Theorem 2.14 - If ϕ be a homomorphism of a group G onto or into a group G', and S be a self compressed subset of G with respect to a subset K in G, then its homomorphic image S' is self compressed with respect to K' in G' where $K' = K\phi$.

Proof. Let $y \in K$ be arbitrary, then

$$\begin{aligned} S'_{(y\phi)} &= (y\phi)\,(S\phi)\,(y\phi) \\ &= (y\,S\,y)\,\phi \\ &= (S)\,\phi = S' \end{aligned}$$

$\Longrightarrow$ S' is self compressed with respect to K'

This proves the theorem.

Cor.2.8 - Let G' be a homomorphic image of a group G under a homomorphism ϕ , and let H be a self compressed subgroup of G with respect to a subset K of G, then the homomorphic image of H is self compressed subgroup of G' with respect to Kϕ .

Cor. 2.9 - If ϕ be a homomorphism of a group G onto a group G', the image of a c.s.c-subset (or c.s.c-subgroup) is a c.s.c-subset (or c.s.c-subgroup).

Theorem 2.15 - Let ϕ be a homomorphism of a group G onto a group G' with kernel K, then a subgroup H of G containing K is self compressed with respect to a subset S of G, if and only if the subgroup H' is self compressed with respect to S' where $H' = H\phi$, $S' = S\phi$.

Proof. Necessary part is obvious from corollary 2.8. For sufficiency, consider

$$H' = (s\phi)\, H'(s\phi) \quad \text{where } s\phi \in S',\ s \in S$$

$$= (s\phi)\,(H\phi)\,(s\phi)$$

$$= ((sHs)\phi)\phi^{-1}$$

$\Longrightarrow H = sHs$ since $H \supseteq K$.

This proves the theorem.

Remarks : (i) The converse of the above theorem is not true, in general, for subsets in a group.

(ii) If $S = G$, the theorem holds for completly self compressed subgroups.

(iii) The theorem also implies that if $\{H_i\}$ denotes the collection of all self compressed subgroups of G, containing K, with respect to a subset S of G, then the mapping $H_i \longrightarrow (H_i)\phi$ is 1-1 of $\{H_i\}$ onto the collection of all self compressed subgroups of G' with respect to S' .

Theorem 2.16 - Let ϕ be an isomorphism of a group G onto a group G', then a subset S of G is self compressed with respect to a subset K of G if and only if $(S)\phi = S'$ is self compressed with respect to $(K)\phi$.

Proof. Necessity is obvious from theorem 2.14.

For sufficiency, since

$$S' = (y\phi)\, S'(y\phi) \qquad \forall\, y \in K$$

$$= (y\phi)\,(S\phi)\,(y\phi)$$

$$= (y\,S y)\phi$$

$$\Longrightarrow S = ySy$$

Hence S is self compressed with respect to K.

This proves the theorem.

Cor.2.10 - Let ϕ be an isomorphism of a group G onto a group G', then a subgroup H of G is self compressed with respect to a subset K of G if and only if $H\phi = H'$ is self compressed with respect to $K\phi = K'$.

If $K = G$ then a subgroup (or subset) of G is completly self compressed if and only if its image is completly self compressed in G'.

Theorem 2.17 - Let ϕ be a homomorphism of a group G onto a group G' and S a subset of G, then $(S)\phi$ is completly self compressed if and only if S_x and $S_{x^{-1}}$ both are contained in $S.\mathrm{Ker}\,\phi$ for all $x \in G$.

Proof. (This theorem immediately follows from theorem 1.14)

Theorem 2.18 - Let S_1, S_2 be any two subsets of a group G such that $e \in S_i$, $i = 1,2$, then

$$[S_1S_2]^2 = [S_1^2]\,[S_2^2]$$

Proof. Let $s_i \in S_i$, $i = 1,2$ be arbitrary, then

$$(s_1s_2)^2 = s_1^2 \cdot s_2^2 \in [S_1^2]\,[S_2^2]$$

$$\Longrightarrow [S_1S_2]^2 \subseteq [S_1^2]\,[S_2^2]$$

On the other hand, we have

$$s_1^2,\ s_2^2 \in [S_1 S_2]^2 \quad \text{since } e \in S_i,\ i = 1,2 .$$

$$\Longrightarrow [S_1^2]\,[S_2^2] \subseteq [S_1S_2]^2$$

Hence

$$[S_1 S_2]^2 = [S_1^2]\,[S_2^2]$$

Thus the proof is complete.

Cor.2.11- If G be any group and $\{S_i\}$, the family of all subsets of G such that $e \in S_i$, then the mapping $S_i \longrightarrow [S_i^2]$ of the semigroup $\{S_i\}$ onto the semigroup of smallest self compressed subgroups with respect to S_i's where $[S_i^2]$ is the smallest self compressed subgroup with respect to S_i, is a homomorphism and its kernel is the subfamily of the subsets S_i's having their elements ($\neq e$) only of order 2.

(Proof is obvious in view of Theorem 2.18)

Theorem 2.19 - Let H_i, H_i', $i = 1,2$ are subgroups of a group G and H_i' is a c.s.c-subgroup of H_i then $H_1'(H_1 \cap H_2')$ is c.s.c-subgroup of $H_1'(H_1 \cap H_2)$ and $H_2'(H_1' \cap H_2)$ is c.s.c-subgroup of $H_2'(H_1 \cap H_2)$.

Proof. We define

$$K = H_1 \cap H_2, \quad K' = H_1 \cap H_2'$$

Evidently $H_1'(H_1 \cap H_2) = H_1'K$ and $H_1'(H_1 \cap H_2') = H_1'K'$ are subgroups of G, and also

$$H_1'(H_1 \cap H_2') \subseteq H_1'(H_1 \cap H_2)$$

Every element of $H_1'K$ is of the form $h_1'k$ where $h_1' \in H_1'$ and $k \in K$, hence since

$$H_1'(H_1 \cap H_2) \subseteq H_1(H_1 \cap H_2) \subseteq H_1 H_1 = H_1$$

$$\Longrightarrow \quad h_1' k \in H_1$$

Also as H_1' is c.s.c-subgroup of H_1, we have

$$h_1' k.\ H_1' (H_1 \cap H_2') .\ h_1' k = (h_1' k.H_1' , h_1' k)(H_1 \cap H_2')$$
$$= H_1'(H_1 \cap H_2')$$

$\Longrightarrow$ $H_1'(H_1 \cap H_2')$ is c.s.c-subgroup of $H_1'(H_1 \cap H_2)$.

Similarily we can show that $H_2'(H_1' \cap H_2)$ is a c.s.c-subgroup of $H_2' (H_1 \cap H_2)$

This completes the theorem.

Note : We note that by Zassenhans Lemma (since G is abelian),

$$\frac{H_1'(H_1 \cap H_2)}{H_1'(H_1 \cap H_2')} = \frac{H_2'(H_1 \cap H_2)}{H_2'(H_1' \cap H_2)}$$

8. Self Compressed Subsets In Direct Products:

The theorems which we prove below are related to the properties of self compressed subsets in relation to internal and external direct products. Actually, we show that the property of self compression of a subset in a direct product is translated to its direct factors.

Theorem 2.20 - Let $G = \overset{n}{\underset{i=1}{X}} G_i$ where G_i's are subgroups of G, and let ϵ_i's be the projections determined by this direct decomposition. Then if a subset S of G is self compressed with respect to a subset K of G, $S\,\epsilon_i$ is self compressed with respect to $K\epsilon_i$, the converse, however, is true if $(S\epsilon_1)(S\epsilon_2)\ldots(S\epsilon_n) = S$.

Proof. Let S be self compressed with respect to K, then for any $k \in K$,

$$kSk = S$$

$$\Longrightarrow (kSk)\epsilon_i = S\epsilon_i \quad \text{for } i = 1,2,\ldots, n .$$

$$\Longrightarrow S\epsilon_i = (k\epsilon_i)(S\epsilon_i)(k\epsilon_i)$$

Thus $S\epsilon_i$ is self compressed with respect to $k\epsilon_i$ and hence with respect to $K\epsilon_i$.

Conversely, if

$$(k\epsilon_i)(S\epsilon_i)(k\epsilon_i) = (S\epsilon_i) \text{ for all } k \in K \text{ and}$$

$i = 1,2,\ldots n$. Then for a fixed $k \in K$,

$$(k\epsilon_1)(S\epsilon_1)(k\epsilon_1)\ldots(k\epsilon_n)(S\epsilon_n)(k\epsilon_n) = (S\epsilon_1)(S\epsilon_2)\ldots(S\epsilon_n)$$

$$\Longrightarrow (k\epsilon_1)(k\epsilon_2)\ldots(k\epsilon_n)\big((S\epsilon_1)\ldots(S\epsilon_n)\big)(k\epsilon_1)\ldots(k\epsilon_n) = (S\epsilon_1)\ldots(S\epsilon_n)$$

$$\Longrightarrow kSk = S \text{ since } \sum_{i=1}^{n} \epsilon_i = 1 \text{ and } (S\epsilon_1)\ldots(S\epsilon_n) = S$$

Hence the proof is complete.

Remark : If $K = G$, the above theorem holds true for c.s.c-subsets. In case of external direct product the theorem assumes the following shape : -

Theorem 2.21 - Let G_i, $i = 1,2, \ldots, n$ be n groups. Then if a subset S of $G = \overset{n}{\underset{i=1}{X}} G_i$ is self compressed with respect to a subset K of G, the component of S in G_i is self compressed with respect to the component of K in G_i for every i. However, the converse is true, if S is the cartesian product of its own components.

Proof. Let S be self compressed with respect to K, then for any $k \in K$,

$$S_k = S$$

$$\Longrightarrow k\,s\,k = s' \in S$$

We define

$$S_i = \text{component of } S \text{ in } G_i$$

$$K_i = \text{component of } K \text{ in } G_i$$

Let $s = (s_1, s_2, \ldots, s_n)$ and $k = (k_1, k_2, \ldots, k_n)$ be elements of S and K respectively denoted as (s_i) and (k_i). Then we have for arbitrary s and fixed k,

$$k\ s\ k = s' \in S$$

$$\Longrightarrow (k_i)(s_i)(k_i) = (s_i') \quad \text{where} \quad s' = (s_i')$$

$$\Longrightarrow (k_i s_i k_i) = (s_i')$$

$$\Longrightarrow k_i s_i k_i = s_i' \quad \text{for all} \quad i = 1,2, \ldots n$$

$$\Longrightarrow k_i\ S_i\ k_i \subseteq S_i$$

Next, since

$$S_k = S$$

$$\implies S = S_{k^{-1}}$$

We have, as above

$$k_i^{-1} S_i k_i^{-1} \subseteq S_i \quad \text{since } k^{-1} = (k_i^{-1})$$

$$\implies S_i \subseteq k_i S_i k_i$$

Hence

$$S_i = k_i S_i k_i$$

Thus S_i is self compressed with respect to k_i and hence with respect to K_i. Conversely, let S_i be self compressed with respect to K_i for every i, and $S = S_1 \times S_2 \times \dots \times S_n$, then for any $s = (s_i) \in S$ and $k = (k_i) \in K$,

$$k\,sk = (k_i)(s_i)(k_i)$$

$$= (k_i\, s_i k_i)$$

$$= (s_i'') \quad \text{where} \quad s_i'' \in S_i$$

$$\implies ksk \in S \quad \text{since} \quad S_1 \times S_2 \times \dots \times S_n = S$$

$$\implies k\,Sk \subseteq S$$

Finally, since

$$(S_i)_{k_i} = S_i \quad \text{for every} \quad k_i \in K_i$$

$$\Longrightarrow \quad S_i = (S_i)_{k_i^{-1}}$$

Considering, $k^{-1} s k^{-1}$, we can show similarily that

$$k^{-1} S k^{-1} \subseteq S$$

$$\Longrightarrow \quad S \subseteq k S k$$

Consequently

$$S = k S k$$

Hence the theorem is proved.

Remark : If $K = G$, the above theorem holds for c.s.c. subsets, and it can also be extended for complete direct products of arbitrary family of groups.

9. Theorems True In Non-Abelian Groups.

All the results, in this chapter, have been proved for abelian groups only, but it can be easily verified that the theorems, 2.1, 2.2, 2.5, 2.6, 2.14, 2.15, 2.16, 2.17, 2.20 and 2.21 are also true for non-abelian groups with no change in arguments in the proofs already supplied.

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CHAPTER - THREE

SMALLEST C.S.C - SUBGROUP AND C -SIMPLE GROUPS

1. Introduction: This chapter is devoted to the study of smallest c.s.c-subgroup of a group denoted as G^* and to groups which contain no c.s.c-subgroup other than itself designated as C-simple groups. We find that the smallest c.s.c-subgroup of a group G is just the subgroup generated by squares of elements of G and is isomorphic to the factor group of the group with respect to the subgroup of all elements of order two in the group. The smallest c.s.c-subgroup of a direct product comes out to be the direct product of smallest c.s.c-subgroups of the direct factors. We investigate that the index of the smallest c.s.c-subgroup in a group, having basis, is a power of 2, actually $2^{r_o(G)+r_2(G)}$ where $r_o(G)$ is the torsion free rank and $r_2(G)$ is 2-rank of G. We prove that the formation of smallest c.s.c-subgroup i.e. the star operation is a homomorphism of the semigroup of all subgroups of a group onto the semigroup of all subgroups of its smallest c.s.c-subgroup, and introduce a notion of essential equivalence between the subgroups of a group. This notion illuminates the study of smallest c.s.c-subgroup in an interesting manner. We use it to show that any two subgroups of a group have the same smallest c.s.c-subgroup if and only if they are essentially equal. We have investigated conditions in which the smallest c.s.c-subgroup of a group becomes cyclic and have pointed out that a group whose smallest c.s.c-subgroup is cyclic, is also cyclic if the smallest c.s.c-subgroup contains the subgroup of all elements

of order 2 in the group. It is interesting to note that the isomorphic groups have isomorphic smallest c.s.c-subgroups but not conversely in general, however, if a group G is isomorphic to a subgroup of a group G_1 then G^* is isomorphic to G_1^* under the same isomorphism, if and only if the isomorphic image of G be essentially equal to G_1. Further, we discuss a condition under which intersection of a subgroup of a group with its smallest c.s.c-subgroup coincides with the smallest c.s.c-subgroup of the subgroup; and find out that smallest c.s.c-subgroup of the anticenter of a group is the anticenter of the smallest c.s.c-subgroup of the group.

Finally, we prove an important theorem on the relation between ranks of a group and of its smallest c.s.c-subgroup showing that they differ by their 2-ranks, and further evaluate the difference in terms of power of elements of order 2. We conclude our discussions with the properties of groups having identical smallest c.s.c-subgroups which we call C-simple groups. Here we show that this class is a subclass of periodic groups, whose elements are of odd order only.

2. Structure Of Smallest C.S.C-Subgroup :

We now prove some theorems which enlighten us about the structure of this important subgroup. We first observe that G^*, the smallest c.s.c-subgroup of a group G is the collection of squares of all the elements of G, and point out that this is a homomorphic image of G with kernel the subgroup of all elements of order two in G. Also, further, every subgroup of the subgroup G^* is actually the smallest c.s.c-subgroup of some subgroup of G. Finally it is important to note that the smallest c.s.c-subgroup of a direct product is the direct product of the smallest c.s.c-subgroups of its direct factors.

Theorem 3.1 - Let G be any group then the set $G^* = \{g^2 \mid g \in G\}$ is a c.s.c-subgroup of G. The subgroup G^* is the smallest subgroup of this type and is unique.

Proof. Evidently, G^* is a subgroup of G and further from cor.2.3, G^* is a c.s.c-subgroup of G. Also G^* is a smallest c.s.c-subgroup of G, for let H be any c.s.c-subgroup of G, we have again from cor.2.3

$$g^2 \in H \quad \text{for every } g \in G$$

$$\Longrightarrow G^* \subseteq H$$

For uniqueness, let H' be a smallest c.s.c-subgroup of G, then since H' is a c.s.c-subgroup of G, we have

$$G^* \subseteq H'$$

$$\Longrightarrow H' = G^*$$

This shows that G^* is unique and the theorem is completly proved.

Cor.3.1 - A subgroup H of a group G is completly self compressed if and only if $H \supseteq G^*$.

(Proof is immediate in view of Cor 2.8)

Theorem 3.2 - The smallest c.s.c-subgroup G^* of a group G is a homomorphic image of the group G with kernel of homomorphism being O_2, the subgroup of all elements of order 2 in G.

Proof. Define a mapping

$$\phi : g \longrightarrow g^2$$

of G onto G^*.

Evidently, ϕ is single valued, ϕ is a homomorphism, since for any $g_1, g_2 \in G$.

$$\begin{aligned}(g_1g_2)\phi &= (g_1g_2)^2 \\ &= g_1^2\, g_2^2 \\ &= (g_1)\phi . (g_2)\phi\end{aligned}$$

The kernel is the set of all elements $g \in G$ for which

$$(g)\phi = e$$

i.e. $$g^2 = e$$

$$\Longrightarrow \text{Kernel of } \phi \text{ is } O_2$$

This proves the theorem.

Cor.3.2 - A group G is isomorphic to G^* if and only if there is no element ($\neq e$) of order 2 in G.

The corollary implies that if G is torsion free, G^* is isomorphic to G. In particular, if G be finite group, we have

Cor.3.3 - A finite group G contains G^* as a proper c.s.c-subgroup if and only if order of G is even but will coincide with G if order of G be odd.

Theorem 3.3 - If G be any group then to every subgroup H' of its smallest c.s.c-subgroup G^*, there exists a subgroup H of G such that $H^* = H'$.

Proof : Given any subgroup H' of G^*, let us define a set

$$H = \{h \mid h \in G \text{ such that } h^2 \in H'\}$$

Since H' is a subgroup, we have for any $h_1, h_2 \in H$

$$\begin{aligned}(h_1 h_2^{-1})^2 &= h_1^2 (h_2^{-1})^2 \\ &= h_1^2 (h_2^2)^{-1} \in H' \\ \Longrightarrow\ & h_1 h_2^{-1} \in H\end{aligned}$$

Hence H is a subgroup of G.

Now, evidently

$$H^* \subseteq H'$$

Also since $H' \subseteq G^*$, to every element $h' \in H'$, there exists an element $g \in H \subseteq G$ such that

$$h' = g^2$$

$$\Longrightarrow H' \subseteq H^*$$

Consequently,

$$H^* = H'$$

This proves the theorem.

Note : It can be easily seen that in the above theorem the subgroup H of G is the largest subgroup for which $H^* = H'$.

Theorem 3.4 - If G be any cyclic group generated by an element a, then its smallest c.s.c-subgroup G^* is the subgroup generated by a^2.

Proof. Firstly, it is clear that

$$a^2 \in G^*$$

$$\Longrightarrow [a^2] \subseteq G^*.$$

Secondly, if a^n be any element of G,

$$(a^n)^2 = (a^2)^n \in [a^2]$$

$$\Longrightarrow G^* \subseteq [a^2]$$

Hence,

$$G^* = [a^2]$$

Theorem 3.5 - If a group G be a direct product $\overset{n}{\underset{i=1}{X}} G_i$ of its subgroups G_i's, then

$$G^* = G_1^* \times G_2^* \times \ldots \times G_n^*$$

Proof. Evidently,

$$G_i^* \subseteq G^* \quad \text{for all } i$$

$$\Longrightarrow G_1^* \times G_2^* \times \ldots \times G_n^* \subseteq G^*$$

on the other hand, for any $g \in G$, let

$$g = g_1 \cdot g_2 \ldots\ldots g_n \quad \text{where } g_i \in G_i,\ i = 1, 2, \ldots\ldots n.$$

$$\Longrightarrow g^2 = (g_1 \cdot g_2 \ldots g_n)^2$$

$$= g_1^2 \cdot g_2^2 \ldots\ldots g_n^2 \in G_1^* \times G_2^* \ldots \times G_n^*$$

$$\Longrightarrow G^* \subseteq G_1^* \times G_2^* \times \ldots \times G_n^*$$

Hence,

$$G^* = G_1^* \times G_2^* \times \ldots\ldots \times G_n^*$$

This completes the theorem.

Cor.3.4 - If a group G has a basis $\{a_\alpha\}_{\alpha \in A}$ where A is an index set then $G^* = \underset{\alpha \in A}{\times} [a_\alpha^2]$.

(Proof is immediate in view of theorems 3.4 and 3.5)

3. Index Of G^* In G :

In the following, we determine $[G : G^*]$ the index in G of the smallest c.s.c-subgroup G^* of a group G having a basis and formulate it in terms of torsion free rank and 2-rank of the group.

Def.3.1 - The non-identity elements $a_1, a_2, \ldots, a_k$ of the group G are called linearly independent, or briefly, independent, if any relation

$$a_1^{n_1} . a_2^{n_2} \ldots\ldots . a_k^{n_k} = e \qquad (n_i \in I)$$

implies

$$a_1^{n_1} = a_2^{n_2} \ldots\ldots = a_k^{n_k} = e$$

i.e. $n_i = 0$ if $O(a_i) = \infty$, and $O(a_i) \mid n_i$ if $O(a_i) < \infty$. In the contrary case, they are called dependent.

Def. 3.2 - The cardinal number of a maximal independent set in a group G containing merely elements of order ∞ is the torsion free rank $r_o(G)$ of G. For any prime p, the p-rank $r_p(G)$ of G is the cardinal number of a maximal independent set in G containing only the elements of orders of powers of p. Now, we prove an important theorem to achieve our end:

Theorem 3.6 - Let G be a finitely generated group and $L = \{a_1, a_2, \ldots, a_m\}$ be the set of all elements of order infinity and of $2^k (k \geq 1)$ in a basis of G containing elements of infinite and / or prime power order, then if $L \neq \phi$, G^* is properly contained in G. Further if $H = [G^*, a_1, a_2, \ldots, a_\ell]$, the subgroup generated by G^* and a_i's such that $a_i \in L$ where $o \leq \ell \leq m$. Then

$$[H : G^*] = 2^\ell .$$

Proof. Let $B = \{a_1, a_2, \ldots, a_n\}$ be a basis of G under consideration then, by a well known theorem

$G = [a_1] \times [a_2] \times \ldots \times [a_n]$ where $[a_i]$ is the cyclic group generated by a_i.

Also, by Cor 3.4, we have

$$G^* = [a_1^2] \times [a_2^2] \times \ldots \times [a_n^2]$$

Evidently $[a_i^2] = [a_i]$ only if $O(a_i) = p^k (k \geq 1)$ where p be an odd prime, hence if $L \neq \phi$ it is clear that

$$G^* \subset G$$

Now we assert that for any $g \in G$ and $a_i \in L$, $g^2 \neq a_i$ Since, otherwise

$$g = a_1^{\alpha_1} \; a_2^{\alpha_2} \ldots\ldots\ldots a_n^{\alpha_n}$$

$$\Longrightarrow g^2 = a_1^{2\alpha_1} \; a_2^{2\alpha_2} \ldots\ldots\ldots a_n^{2\alpha_n} = a_i$$

$$\Longrightarrow a_1^{2\alpha_1} \; a_2^{2\alpha_2} \ldots\ldots\ldots a_i^{2\alpha_i - 1} \ldots\ldots a_n^{2\alpha_n} = e$$

$$\Longrightarrow a_1^{2\alpha_1} = a_2^{2\alpha_2} = \ldots = a_i^{2\alpha_i - 1} = \ldots = a_n^{2\alpha_n} = e$$

since B is a linearly independent set.

$\Longrightarrow 2\alpha_i - 1$, the power of a_i, is either zero or divisible by 2^k for some $k \geq 1$.

A contradiction that α_i is integral, hence our assertion follows. It is therefore clear that,

$$g^2 \neq a_{1'} \cdot a_{2'} \ \ldots \ a_{h'} \quad \text{for any } g \in G \text{ and } a_{i'} \in L$$

We now put $L' = \{a_1, a_2, \ldots a_\ell\}$ and define

$$K_1 = \text{set of all cosets } a_i G^*, \ a_i \in L'$$

$$K_2 = \text{set of all cosets } a_i a_j G^*, \ a_i, a_j \in L'$$

....

$$K_\ell = \text{The coset } a_1 \cdot a_2 \ldots a_\ell G^*$$

We assert that $H = G^* \cup K_1 \cup K_2 \cup \ldots \cup K_\ell$

Clearly, we have

$$G^* \cup K_1 \cup K_2 \cup \ldots \cup K_\ell \subseteq H$$

Also, since for any $a_i \in L'$

$$(a_i G^*)^{-1} = a_i^{-1} G^*$$
$$= a_i G^*, \text{ as } a_i^2 \in G^*$$

and

$$(a_i G^*)^n = a_i^n G^*$$
$$= G^* \text{ or } a_i G^* \text{ according as } n \text{ is even or odd.}$$

Hence every element $a_1^{\epsilon_1} a_2^{\epsilon_2} \ldots a_\ell^{\epsilon_\ell} G^*$ of H/G^* on the basis of above considerations is of the form

$$a_{1'} a_{2'} \ldots a_{k'} G^*$$

where $a_{i'}$'s are all distinct elements of L'

$$\Longrightarrow \quad H \subseteq G^* \cup K_1 \cup K_2 \cup \dots \cup K_\ell$$

Consequently,

$$H = G^* \cup K_1 \cup K_2 \cup \dots \cup K_\ell$$

$$\Longrightarrow [H : G^*] \leq ({}^{\ell}c_0 + {}^{\ell}c_1 + \dots + {}^{\ell}c_\ell) = 2^\ell$$

Finally, if for any two different subsets $S' = \{a_{1'}, a_{2'}, \dots, a_{i'}\}$ and $S'' = \{a_{1''}, a_{2''}, \dots, a_{j''}\}$ in L' we get

$$a_{1'}\, a_{2'} \dots a_{i'} G^* = a_{1''}\, a_{2''} \dots a_{j''} G^*$$

$$\Longrightarrow \quad a_{1'}\, a_{2'} \dots a_{i'} = a_{1''}\, a_{2''} \dots a_{j''}\, g^2 \text{ where } g \in G$$

Now if $g = a_1^{\beta_1}\; a_2^{\beta_2} \dots a_n^{\beta_n}$,

$$a_{1'}\, a_{2'} \dots a_{i'} = a_{1''}\, a_{2''} \dots a_{j''}\; a_1^{2\beta_1}\, a_2^{2\beta_2} \dots a_n^{2\beta_n}$$

$$\Longrightarrow a_1^{\beta_{1'}}\, a_2^{\beta_{2'}} \dots a_{h'}^{2\beta_{h'}-1} \dots a_n^{\beta_{n'}} = e \text{ if } a_{h'} \in S' \text{ and } S''$$

$$\Longrightarrow a_h^{2\beta_{h'}-1} = e \quad \text{since B is a linearly independent set.}$$

A contradiction that $\beta_{h'}$ is integral.

Therefore, the two cosets in question cannot be equal and hence are distinct.

$$\Longrightarrow [H : G^*] \nless 2^{\ell} .$$

Thus we prove

$$[H : G^*] = 2^{\ell}.$$

Cor.3.5 - If G be a finitely generated group then,

$$[G : G^*] = 2^{r_0(G) + r_2(G)}$$ where $r_0(G)$ = torsion free rank of G, $r_2(G)$ = 2 - rank of G.

(The proof follows immediately if L' = L since then $[G : G^*] = 2^m$)

Remark : The theorem is proved above for finitely generated groups, but if G be any group having a basis, the result holds true. In case, L is infinite, $[G : G^*]$ is infinite. In particular, if G = [a] be a cyclic group, then $[G : G^*] = 2$, if G be of infinite or even order and is equal to 1 if otherwise.

4. Rank Of A Group And Of Its Smallest C.S.C-Subgroup:

We shall show that ranks of a group and of its smallest c.s.c-subgroup differ by the difference of the 2-ranks of the group and of its smallest c.s.c-subgroup. To get at this, we shall first prove some lemmas.

Lemma 1- In a group G, if a set of elements $\{g_i\}_{i=1}^{n}$, containing no element of order 2 is linearly independent implies $\{g_i^2\}_{i=1}^{n}$ is linearly independent in G^*.

Proof. Let for a system of integers $\{\alpha_i\}_{i=1}^{n}$,

$$(g_1^2)^{\alpha_1}(g_2^2)^{\alpha_2}\cdots\cdots\cdots(g_n^2)^{\alpha_n} = e$$

$$\Longrightarrow g_1^{2\alpha_1}\, g_2^{2\alpha_2}\cdots\cdots\cdots g_n^{2\alpha_n} = e$$

$$\Longrightarrow g_i^{2\alpha_i} = e \quad \text{since } \{g_i\}_{i=1}^{n} \text{ is linearly independent.}$$

$$\Longrightarrow (g_i^2)^{\alpha_i} = e \quad \text{for all } i = 1,2,\ldots\ldots n.$$

Hence, since $\{g_i^2\}_{i=1}^{n}$ is a system of elements $(\neq e)$, it is linearly independent.

Lemma 2 - In a group G, $r_o(G) = r_o(G^*)$ and also for any prime $p(\neq 2)$, $r_p(G) = r_p(G^*)$.

Proof. For the first part $r_o(G) = r_o(G^*)$, let $\{g_\alpha\}_{\alpha\in A}$ be any maximal linearly independent system of elements of infinite order in G. Then $\{g_\alpha^2\}_{\alpha\in A}$ is a system of elements of order infinity in G^*. Since linear independence is a property of finite character, it follows from lemma 1, that $\{g_\alpha^2\}_{\alpha\in A}$ is a linearly independent system of elements in G^*. Hence

$$r_o(G) \leq r_o(G^*)$$

But clearly, since $G^* \subseteq G$,

$$r_o(G^*) \leq r_o(G)$$

Consequently

$$r_o(G) = r_o(G^*)$$

For the second part $r_p(G) = r_p(G^*)$ for all primes $p(\neq 2)$, let $\{g_\mu\}_{\mu \in M}$ be a maximal linearly independent system of elements in G containing elements of orders of powers of a prime $p(\neq 2)$. Again, it is evident from lemma 1, that $\{g_\mu^2\}_{\mu \in M}$ is a linearly independent system of elements in G^* and also it can be checked that orders of elements in it are powers of same prime p.
Hence

$$r_p(G) \leq r_p(G^*)$$

Also, it is clear that

$$r_p(G^*) \leq r_p(G)$$

Thus

$$r_p(G) = r_p(G^*)$$

This completely proves the lemma.

Theorem 3.7 - For any group G, $r(G) + r_2(G^*) = r(G^*) + r_2(G)$.

Proof. We know from [3] theorem 8.2, that

$$r(G^*) = r_o(G^*) + \sum_{p=2,3,5,\ldots} r_p(G^*)$$

$$= r_o(G) + \sum_{p=3,5,\ldots} r_p(G) + r_2(G^*) \text{ by lemma 2.}$$

$$\Longrightarrow \quad r(G^*) = r(G) - r_2(G) + r_2(G^*)$$

$$\Longrightarrow \quad r(G) + r_2(G^*) = r(G^*) + r_2(G)$$

This completes the theorem.

Theorem 3.8 - For any group G,

(i) $r(G^*) = r(G) - \log_2 \dfrac{|G[2]|}{|G^*[2]|}$ if $r_2(G) < \infty$.

(ii) $r(G) - r(G^*) = |G[2]| - |G^*[2]|$ if $r_2(G) = \infty$.

Proof.(1) We know from theorem 3.7, that

$$r(G) - r(G^*) = r_2(G) - r_2(G^*)$$

Now

$r_2(G) = r(G_2)$ where G_2 is the 2-component of the maximal torsion subgroup of G.

$$= r(S(G_2))$$

$$= r(G_2[2])$$

$$= r(G[2])$$

From [3] p.33

$|S(G_2)| = 2^k$ where $K = r(S(G_2))$

$$\Longrightarrow |G[2]| = 2^{r_2(G)}$$

$$\Longrightarrow r_2(G) = \log_2 |G[2]|$$

Similarily

$$r_2(G^*) = \log_2 |G^*[2]|$$

Hence

$$r(G) - r(G^*) = \log_2 \frac{|G[2]|}{|G^*[2]|}$$

This proves (i)

(ii) If $r_2(G) = \infty$,

$$r(G_2) = r(S(G_2)) = \infty$$

$$\Longrightarrow r(S(G_2)) = |S(G_2)| \quad \text{by [3] p. 33}$$

$$\Longrightarrow r_2(G) = |G[2]|$$

Similarily

$$r_2(G^*) = |G^*[2]|$$

$$\Longrightarrow r(G) - r(G^*) = |G[2]| - |G^*[2]|$$

This completes the proof.

Cor. 3.6 - In a group G, $r(G) = r(G^*)$ if and only if $|G[2]| = |G^*[2]|$

5. Smallest C.S.C-Subgroup And Fundamental Mappings:

We observe that any homomorphism of a group onto another group induces a homomorphism of the smallest c.s.c-subgroup of the group onto the smallest c.s.c-subgroup of the image group, and further establish that the formation of smallest c.s.c-subgroup is a homomorphism of the semigroup of all subgroups of a group G onto the semigroup of all subgroups of G^*.

Theorem 3.9 - If ϕ be a homomorphism of a group G onto a group G_1 then the smallest c.s.c-subgroup of G_1 is the homomorphic image of the smallest c.s.c-subgroup of G under ϕ.

Proof. From Theorem 3.1, we know that G^* is a c.s.c-subgroup of G and hence from cor. 2.9, $(G^*)\phi$ is c.s.c-subgroup of G_1. Thus by Cor. 3.1,

$$(G^*)\phi \supseteq G_1^*$$

Again from theorem 3.1, for any $g \in G$

$$(g^2)\phi = (g\,\phi)^2 \in G_1^*$$

$$\Longrightarrow (G^*)\phi \subseteq G_1^*$$

Hence,

$$(G^*)\,\phi = G_1^*$$

This proves the theorem.

Theorem 3.10 - If H_1, H_2 be any two subgroups of a group G then,

$$[H_1 \cdot H_2]^* = H_1^* \cdot H_2^*$$

Proof. Clearly

$$H_i^* \subseteq [H_1 \cdot H_2]^* \quad \text{for} \quad i = 1,2$$

$$\Longrightarrow H_1^* \cdot H_2^* \subseteq [H_1 \cdot H_2]^*$$

On the other hand, if $h_i \in H_i$, $i = 1,2$ then $(h_1 h_2)^2 \in [H_1 \cdot H_2]^*$.

Now

$$(h_1 h_2)^2 = h_1^2\, h_2^2 \in H_1^* \cdot H_2^*$$

$$\Longrightarrow [H_1 \cdot H_2]^* \subseteq H_1^* \cdot H_2^*$$

Consequently

$$[H_1 \cdot H_2]^* = H_1^* \cdot H_2^*$$

This proves the theorem.

Theorem 3.11 - If $\{H_i\}$ denotes the semigroup of all subgroups of a group G then,

$$H_i \longrightarrow H_i^*$$

is a homomorphism of $\{H_i\}$ onto the semigroup of all subgroups of G^*.

Proof. (It immeidatly follows from theorems 3.3 and 3.10).

6. Essential Equality In Subgroups And Their Smallest C.S.C-Subgroups:

We define here a new concept of essential equality between subgroups of a group and establish that any two subgroups of a group have the same smallest c.s.c-subgroup if and only if they are essentially equal and note that the 'essential equality' between subgroups of a group is an equivalence relation. This fact enables us to partition the family of all subgroups of a group into disjoint subclasses such that all subgroup in one class have the same smallest c.s.c-subgroup. Further, we also observe that product of any two such partition classes is in a partition class. We show also that any two subgroups of a group have the same smallest c.s.c-subgroup if and only if so do their isomorphic images; in case of homomorphism however, the c.s.c-subgroups of given subgroups should contain the kernel.

Def. 3.3 - Any two subgroups H_1, H_2 of a group G are said to be essentially equal if

$$H_1 O_2 = H_2 O_2$$

where O_2 is the subgroup of all elements of order 2 in G, in symbols, $H_1 \overset{=}{\longleftrightarrow} H_2$.

The following observations can be easily checked:

(i) If H_1, H_2 be any two subgroups of a group G such that $H_i \supseteq O_2$ for $i = 1,2$ then,

$$H_1 \overset{=}{\longleftrightarrow} H_2 \text{ if and only if } H_1 = H_2$$

(ii) For every subgroup H of a group G, $H \overset{=}{\longleftrightarrow} H O_2'$ where O_2' is a subgroup of O_2.

(iii) In a group G, O_2 and any of its subgroup are essentially equal.

(iv) Essential equality is an equivalence relation in the family of all subgroups of a group.

Theorem 3.12 - Let H_1, H_2 be any two subgroups of a group G then,

$$H_1^* = H_2^* \text{ if and only if } H_1 \overset{\div}{\longleftrightarrow} H_2.$$

Proof. Let $H_1^* = H_2^*$, then to any $h_1 \in H_1$ there exists $h_2 \in H_2$ such that

$$h_1^2 = h_2^2$$

$\Longrightarrow$ $h_1 h_2^{-1} \in O_2$ where O_2 is the subgroup of all elements of order 2 in G.

$$\Longrightarrow h_1 \in h_2 O_2$$

$$\Longrightarrow H_1 O_2 = H_2 O_2$$

$$\Longrightarrow H_1 \overset{\div}{\longleftrightarrow} H_2$$

Conversely, let

$$H_1 \overset{\div}{\longleftrightarrow} H_2$$

$$\Longrightarrow H_1 O_2 = H_2 O_2$$

$$\Longrightarrow (H_1 O_2)^* = (H_2 O_2)^*$$

$$\Longrightarrow H_1^* O_2^* = H_2^* O_2^* \qquad [\text{ Theorem } 3.10]$$

$$\Longrightarrow H_1^* = H_2^* \quad \text{since} \quad O_2^* = e$$

$$\Longrightarrow H_i^* = H_i'^*$$

$$\Longrightarrow H_1^* . H_2^* = H_1'^* \, H_2'^*$$

$$\Longrightarrow [H_1 . H_2]^* = [H_1' . H_2']^* \quad \text{(Theorem 3.10)}$$

$$\Longrightarrow H_1 H_2 \overset{=}{\longleftrightarrow} H_1' H_2' \text{ since } [H_1 . H_2] = H_1 H_2,$$

$$[H_1' . H_2'] = H_1' . H_2'$$

This completes the proof.

Theorem 3.14 - If H_1, H_2 be two subgroups of a group G and ϕ be an isomorphism of G onto a group G', then

$$H_1 \overset{=}{\longleftrightarrow} H_2 \text{ if and only if } (H_1)\phi \overset{=}{\longleftrightarrow} (H_2)\phi$$

Proof. By theorem 3.12,

$$H_1 \overset{=}{\longleftrightarrow} H_2$$

$$\Longleftrightarrow H_1^* = H_2^*$$

$$\Longleftrightarrow (H_1^*)\phi = (H_2^*)\phi$$

$$\Longleftrightarrow (H_1\phi)^* = (H_2\phi)^* \quad \text{(Theorem 3.9)}$$

$$\Longleftrightarrow (H_1)\phi \overset{=}{\longleftrightarrow} (H_2)\phi \quad \text{(Theorem 3.12)}$$

Hence the theorem is complete.

Cor.3.9 - If H_1, H_2 be any two subgroups of a group G and ϕ be an isomorphism of G onto a group G', then

$$H_1^* = H_2^* \text{ if and only if } (H_1\phi)^* = (H_2\phi)^*.$$

Theorem 3.15 - If H_1, H_2 be any two subgroups of a group G and ϕ be a homomorphism of G onto a group G' with kernel K, then if H_1^* and H_2^* contain K.

$$H_1 \overset{=}{\underset{-}{\longleftrightarrow}} H_2 \text{ if and only if } (H_1)\phi \overset{=}{\underset{-}{\longleftrightarrow}} (H_2)\phi$$

Proof. If $H_1 \overset{=}{\underset{-}{\longleftrightarrow}} H_2 \Longrightarrow (H_1)\phi \overset{=}{\underset{-}{\longleftrightarrow}} (H_2)\phi$ follows as in theorem 3.14.

For converse, if

$$(H_1)\phi \longleftrightarrow (H_2)\phi$$

$$\Longrightarrow \quad (H_1\phi)^* = (H_2\phi)^*$$

$$\Longrightarrow \quad (H_1^*)\phi = (H_2^*)\phi$$

$$\Longrightarrow \quad H_1^* = H_2^* \quad \text{since} \quad K \subseteq H_1^*$$

$$\Longrightarrow \quad H_1 \overset{=}{\underset{-}{\longleftrightarrow}} H_2$$

Hence the theorem is complete.

Cor. 3.10 - If H_1, H_2 be any two subgroups of a group G and ϕ be a homomorphism of G onto G', then

$$H_1^* = H_2^* \text{ if and only if } (H_1\phi)^* = (H_2\phi)^* \text{ where } H_1^* \supseteq \text{Ker. } \phi.$$

7. G^* As A Cyclic Subgroup:

We investigate below the circumstances in which the smallest c.s.c-subgroup of a group becomes cyclic and find out a condition under which a group is cyclic if its smallest c.s.c-subgroup is cyclic.

Theorem 3.16 - If a group G is cyclic, then G^* is cyclic but conversely G is cyclic if G^* is cyclic and contains O_2, the subgroup of all elements of order 2 in G.

Proof. Firstly, if G be cyclic, then clearly G^* is cyclic. Conversely if G^* be cyclic, let

$$G^* = [g_1^2] \quad \text{where} \quad g_1 \in G$$

Then for any $g \in G$, since $g^2 \in G^*$

$$g^2 = (g_1^2)^i \text{ where } i \in I$$

$$\Longrightarrow (g\, g_1^{-i})^2 = e$$

$$\Longrightarrow g g_1^{-i} \in O_2 \subseteq G^*$$

$$\Longrightarrow g \in g_1^i \, [g_1^2] \subseteq [g_1]$$

$$\Longrightarrow G \subseteq [g_1] \subseteq G$$

Hence $G = [g_1]$

This completes the theorem.

Cor 3.11 - If G be a group in which $O_2 = e$, then G is cyclic if and only if G^* is cyclic .

Theorem 3.17 - If G be any finitely generated group, then $G^* (\neq e)$ is cyclic if any basis of G contains only one element of order other than 2.

Proof. (It immediately follows from Cor. 3.4, since

$$G^* = [a_1^2] \times \ldots \times [a_n^2]$$

where $\{a_1, a_2, \ldots a_n\}$ be a basis of G)

Note : The theorem 3.17, also holds for any group G having basis .

8. Relations In G/H, $(G/H)^*$ And G^*/H^* :

We show now an interesting situation where a factor group, its smallest c.s.c-subgroup and the factor group of the corresponding smallest c.s.c-subgroups are all isomorphic.

Theorem 3.18. If H be any subgroup of a group G, then

$$(G/H)^* \cong G^*/H^*$$

Proof. Since every element of $(G/H)^*$ is of the form $g^2 H$. We define a mapping ϕ of $(G/H)^*$ onto G^*/H^* as

$$\phi : g^2 H \longrightarrow g^2 H^*$$

Evidently ϕ is single valued. If $g_1^2 H, g_2^2 H \in (G/H)^*$, then

$$\begin{aligned}((g_1^2 H)(g_2^2 H))\phi &= (g_1^2 g_2^2 H)\phi \\ &= ((g_1 g_2)^2 H)\phi \\ &= (g_1 g_2)^2 H^* \\ &= (g_1^2 H^*)(g_2^2 H^*) \\ &= (g_1^2 H)\phi\, (g_2^2 H)\phi\end{aligned}$$

$\Longrightarrow$ ϕ is a homomorphism

Finally, if

$$(g_1^2 H)\phi = (g_2^2 H)\phi$$

$$\Longrightarrow g_1^2 H^* = g_2^2 H^*$$

$$\Longrightarrow g_1^2 (g_2^2)^{-1} \in H$$

$$\Longrightarrow g_1^2 H = g_2^2 H$$

Hence ϕ is an isomorphism

This proves the theorem.

Theorem 3.19 - If H be any subgroup of a group G, then

$$G/H \simeq G^*/H^*$$

and the kernel of the homomorphism is $H\,O_2/H$ where O_2 is the subgroup of all elements of order 2 in G.

Proof. We define a mapping η of G/H onto G^*/H^* as

$$\eta : gH \longrightarrow g^2 H^*$$

Clearly η is single valued. Further for g_1H, $g_2H \in G/H$

$$\begin{aligned} ((g_1H)(g_2H))\eta &= (g_1g_2H)\eta \\ &= (g_1g_2)^2 H^* \\ &= (g_1^2 H^*)(g_2^2 H^*) \\ &= (g_1H)\eta(g_2H)\eta \end{aligned}$$

$\Longrightarrow$ η is a homomorphism.

Finally, if

$$(gH)\eta = H^*$$

$$\begin{aligned} &\iff g^2 H^* = H^* \\ &\iff g^2 \in H^* \\ &\iff g^2 = h^2 \quad \text{where } h \in H \\ &\iff g h^{-1} \in O_2 \\ &\iff g \in h O_2 \\ &\iff g H \in HO_2 / H \end{aligned}$$

Hence the theorem is complete.

Cor.3.12 - If $H \supseteq O_2$, $G/H \cong G^*/H^*$.

(Proof follows immediatly, since by fundamental theorem of homomorphism $\frac{G/H}{HO_2/H} \cong G^*/H^*$).

Cor. 3.13 - For any subgroup H of a group G, if $H \supseteq O_2$

$$G/H \cong (G/H)^* \cong G^*/H^* .$$

Proof (It immediatly follows from theorem 3.18 and Cor.3.12)

9. Non Isomorphic Groups Having Isomorphic Smallest C.S.C.Subgroups:

Theorem 3.20- Any two isomorphic groups have isomorphic smallest c.s.c- subgroups but not conversely.

Proof. The direct part of the theorem is obvious in view of theorem 3.9. For the converse, consider

$$G = [a,b]; \; a^3 = e, \; b^2 = e, \; ab = ba$$

$$G_1 = [a'], \text{ the cyclic group of order 3.}$$

which give

$$G^* = \{a, a^2, e\}, \quad G_1^* = \{a', a'^2, e\}$$

We observe that G_1^* and G^* being cyclic groups of the same order are isomorphic but G is not isomorphic to G_1 since $O(G) > O(G_1)$. Thus the converse does not hold.

Theorem 3.21 - If ϕ be an isomorphism of a group G into a group G_1, then $G^* \cong G_1^*$ if and only if $(G)\phi \overset{\div}{\longleftrightarrow} G_1$.

Proof. (The proof is immediate in view of theorems 3.9 and 3.12)

Cor. 3.14 - Any two groups whose any two bases under an isomorphism of one group into the other differe only by elements of order 2, have isomorphic smallest c.s.c-subgroups under the same isomorphism.

Proof. Let, for two groups G, G_1 and for an isomorphism ϕ of G into G_1 the condition of the corollary be satisfied, then it is evident that $(G)\phi \overset{\div}{\longleftrightarrow} G_1$, and hence, from the above theorem, the proof follows immediately.

10. $G_1 \cap G^* = G_1^*$ For An Arbitrary Subgroup G_1.

We determine, in this section, a condition for the smallest c.s.c-subgroup of an arbitrary subgroup of a group to be equal to the intersection of the smallest c.s.c-subgroup of the group with the arbitrary subgroup, and further find out under the same condition, the existance of a c.s.c-subgroup of the group corresponding to each c.s.c-subgroup of an arbitrary subgroup.

Theorem 3.22 - If G_1 be any subgroup of a group G, then

$$G_1 \cap G^* = G_1^* \text{ if and only if } (G_1 - G_1^*) \cap G^* = \phi$$

Proof. Let

$$(G_1 - G_1^*) \cap G^* = \phi$$

$$\Longrightarrow \quad G^* \subseteq (G_1 - (G_1 - G_1^*)) = G_1^*$$

But clearly

$$G_1^* \subseteq G^*$$

$$\Longrightarrow \quad G_1^* = G^*$$

Conversely, let $G_1 \cap G^* = G_1^*$. Now suppose $g_1 \in (G_1 - G_1^*)$ such that

$$g_1 = g^2 \in G^*$$

$$\Longrightarrow \quad g_1 \in G_1 \cap G^* = G_1^*$$

A contradiction, hence

$$(G - G_1^*) \cap G^* = \phi$$

Thus the proof is complete.

Theorem 3.23 - If for a subgroup G_1 of a group G, $G_1 \cap G^* = G_1^*$ then for any c.s.c-subgroup $\overline{G_1}$ of G_1 there exists a c.s.c-subgroup $\overline{G}$ of G such that $\overline{G_1} = \overline{G} \cap G_1$.

Proof. We define

$$\bar{G} = [\bar{G}_1, G^*]$$

Evidently, G is a c.s.c-subgroup of G since $\bar{G} \supseteq G^*$. Also

$$\bar{G}_1 \subseteq \bar{G} \cap G_1$$

Now, if $g' \in \bar{G} \cap G_1$, we have

$$g' = g_1 = g^* \bar{g} \text{ where } g_1 \in G_1,\ g^* \in G^*,\ \bar{g} \in \bar{G}_1$$

$$\Longrightarrow \quad g^* = g_1(\bar{g})^{-1} \in G_1$$

$$\Longrightarrow \quad g^* \in G_1^* \quad \text{since} \quad G_1 \cap G^* = G_1^*$$

$$\Longrightarrow \quad g' = g^* \bar{g} \in \bar{G}_1 \quad \text{since } G_1^* \subseteq \bar{G}_1$$

$$\Longrightarrow \quad \bar{G} \cap G_1 \subseteq \bar{G}_1$$

Consequently

$$\bar{G}_1 = \bar{G} \cap G_1$$

Thus the result is established.

11. Groups With Identical Smallest C.S.C-Subgroups:

Def. 3.4 - A group G is called C-simple if $G = G^*$.

Evidently the following are some of the classes of C-simple groups.

(i) Finite groups of odd order

and

(ii) Groups having a basis each of whose element is of odd order.

Theorem 3.24 - The union of an ascending sequence of C-simple groups is itself a C-simple group.

Proof. Let a group G be union of an ascending sequence

$$G_1 \subset G_2 \subset G_3 \subset \ldots\ldots\ldots \subset G_n \subset \ldots\ldots\ldots\ldots$$

of C-simple proper subgroups of G. If G is not C-simple, then there exists a proper c.s.c-subgroup H of G. Now, since $H \subset G$, for some index k,

$$G_k \cap H \neq G_k$$

$$\Longrightarrow \quad G_k^* \neq G_k \quad \text{since } G_k = G_k^* \text{ and } H \supseteq G^* \supseteq G_k^*$$

A contradiction, that G_k is C-simple. Hence the result follows.

Theorem 3.25 - If every subgroup of a group is C-simple then the group is C-simple but not conversely.

Proof. Let

$$G = \left\{ (r_1, r_2, \ldots, r_n, \ldots.) \mid r_i's \in R \right\}$$

be the group of all sequences of rational numbers with respect to addition and suppose

$$H = \left\{ (i_1, i_2, \ldots, i_n, \ldots.) \mid i's \in I \right\}$$

be the subgroup of G consisting of all sequences of integers, then it is evident that

$$G^* = G$$

$$\text{but} \qquad H \neq H^*$$

Conversely, if the condition be satisfied, the proof is trivial.

This completes the theorem.

12. Anticenter And Smallest C.S.C-Subgroup:

This section is of more or less academic interest. Here, we establish that anticenter of the smallest c.s.c-subgroup of a group is the smallest c.s.c-subgroup of the anticenter of the group. To begin with we give the following definitions in case of an arbitrary (abelian or non-abelian) group.

Def. 3.5 - In a group G, the set $R(G) = \{g \mid gh = hg$ for any $h \in G \Longrightarrow g = k^i, h = k^j$ for some $k \in G,\ i,j \in I\}$ is called <u>rim of G.</u>

Def. 3.6 - If G be a group then the subgroup generated by R(G), the rim of G is called 'anticenter' of G and is denoted by AC(G).

Theorem 3.26 - Let G be any group in which O_2, the subgroup of all elements or order 2 in G be identity, then

$$AC(G^*) = (AC(G))^*$$

Proof. Let $g \in AC(G)$ then for any $h \in G$

$$g^2h^2 = h^2g^2$$

$$\Longrightarrow \quad gh = hg$$

$$\Longrightarrow \quad g = g_1^i,\ h = g_1^j\ ,\ i,j \in I,\ g_1 \in G$$

$$\Longrightarrow \quad g^2 = (g_1^2)^i,\ h^2 = (g_1^2)^j \quad \text{where} \quad g_1^2 \in G^*$$

$$\Longrightarrow \quad g^2 \in R(G^*)$$

$$\Longrightarrow \quad g^2 \in AC(G^*)$$

$$\Longrightarrow (AC(G))^* \subseteq AC(G^*)$$

On the other hand, if $g^2 \in AC(G^*)$ then for any $h \in G$, we have

$$gh = hg$$

$$\Longrightarrow g^2h^2 = h^2g^2$$

$$\Longrightarrow g^2 = (g'^2)^i,\ h^2 = (g'^2)^j,\ i,j \in I,\ g'^2 \in G^*$$

$$\Longrightarrow (gg'^{-i})^2 = e,\quad (hg'^{-j})^2 = e$$

$$\Longrightarrow gg'^{-i} = e,\quad hg'^{-j} = e \quad \text{since } O_2 = e$$

$$\Longrightarrow \quad g = g'^i,\ h = g'^j \quad \text{where } g' \in G$$

$$\Longrightarrow \quad g \in AC(G)$$

$$\Longrightarrow \quad g^2 \in (AC(G))^*$$

Hence,

$$AC(G^*) \subseteq (AC(G))^*$$

Consequently,

$$AC(G^*) = (AC(G))^*$$

This completes the theorem.

Cor. 3.15 - In any group G, if $r_2(G) = 0$, $AC(G^*) = (AC(G))^*$ where $r_2(G)$ is the 2-rang of G. In particular if G be torsion free the result holds.

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CHAPTER - FOUR

COMPRESSOR IN AN ABELIAN GROUP

1. **Introduction :** If S be a subset of a group G, then the set of all elements x in G, such that $xS = Sx$ is called the normalizer of S in G. Since the normalizer of any subset of an abelian group is the group itself, the notion when associated with abelian groups reduces to be trivial one. We, therefore, utilise the concept of self compression in defining an analogous notion of 'compressor' of a subset in a group, which plays an important part in abelian groups specially. The set of all those elements of a group G, with respect to which a given set S is self compressed, we call to be the 'compressor of S in G' and denote it by $C_G(S)$. This chapter studies the power of this concept in abelian groups. We first show that the compressor of any subset in a group is a subgroup of the group and find out some fundamental properties such as : 'The compressor of a set and of any of its C-transforms in the same', 'The compressor of a subset and of its compliment is the same', 'Intersection (or union) of the compressors of two subsets is contained in the compressor of the intersection (or union) of those subsets', 'The product of compressors of two subsets is contained in the compressor of product of subsets', 'Compliment of the compressor of a subset is the compliment of the compressor of the compliment of the subset', 'The compressor of any subgroup of index 2, in a group, is the whole group but not conversely', 'Any two finite subgroups containing no elements of order two such that the compressor of one contains the other, coincide with each other' etc. We note that the isomorphic image of the compressor of a subset in a group under an isomorphism of the group is the

compressor of the image subsets in the image group, and also prove that in a direct product the direct product of the compressors of subsets of direct factors in respective direct factors is the compressor of the product of these subsets in the direct product group. We have established that the smallest c.s.c-subgroup of the compressor of a subgroup is contained in the compressor of the smallest c.s.c-subgroup of the subgroup. We further show that in a cyclic group, the compressor of the anticenter of a subgroup coincides with the anticenter of the compressor of the subgroup, but not in general. We have also investigated that any subset is the union of all distinct cosets of the smallest c.s.c-subgroup of its compressor with respect to all elements of the subset, and also determine the conditions under which the compressor of a set in a group given as a union of cosets of a subgroup H of the smallest c.s.c-subgroup of the group coincides with the maximal subgroup of the group, whose smallest c.s.c-subgroup be the given subgroup H. We introduce the concept of c-power of an element of a group in a subgroup of the group and utilise it in obtaining a criterion of equality of a subgroup with its compressor. Finally, since compressor of a subgroup contains the subgroup so the compressor of compressor contains the compressor and in this way we get an ascending chain of subgroups in a group, and obtain the circumstances when it breaks off.

2. Definition And Basic Properties of Compressor:

We define below the notion of compressor of a subset in a group, and give some criteria for an element of a group to belong to the compressor of a finite or infinite subset or of a subgroup. We find out that the compressor of any subset in a group is a subgroup. We also observe that the compressor of a subset and of its compliment in a group is the same, and note that every non identity element of the compressor of a subset of two elements including identity and an element of order other than two is of order two only.

Def. 4.1 - For a subset S of a group G, the set of all elements $x \in G$ such that $xSx = S$ is called 'compressor' of S in G, in symbols $C_G(S)$.

The following observations can be easily checked.

(i) For any subgroup H of G, $H \subseteq C_G(H)$

(ii) The compressor of any element of G is O_2, the subgroup of all elements of order 2 in G.

(iii) A subset S of G is completly self compressed if and only if $C_G(S) = G$.

(iv) The Compressor of G in G is the group G itself.

(v) For any subset S of G, $C_G(S) \supseteq O_2$, the subgroup of all elements of order 2 in G.

(vi) If H be any subgroup of G, which does not contain the subgroup of all elements of order 2 in G, then, for any subset S of G, $C_G(S) \neq H$.

Theorem 4.1 - Let S be a finite subset of a group G, then $x \in C_G(S)$ if and only if to every $s \in S$ there exists $s' \in S$ such that $s = s'_x$

(Proof is immediate from theorem 2.1)

Remark : By our remark in chapter - two after theorem 2.1, it is immediate that the above theorem is not true in general for infinite subsets; however, if instead of S, we take an arbitrary subgroup H of G, then the theorem holds in view of theorem 1.8. We have, in general, the following theorem:

Theorem 4.2 - If S be an arbitrary subset of a group G, then $x \in C_G(S)$ if and only if $x^i \in C_G(S)$ for all $i \in I$.

(Proof follows from theorem 2.2).

Theorem 4.3 - Let S be any subset of a group G, then $C_G(S)$ is a subgroup and is the largest subset with respect to which S is self compressed.

Proof. For any $x, y \in C_G(S)$,

$$xSx = S \quad \text{and} \quad ySy = S$$

$$\begin{aligned} \Longrightarrow \quad xy^{-1}.S.xy^{-1} &= y^{-1}.xSx.\,y^{-1} \\ &= y^{-1}\,S\,y^{-1} \\ &= y^{-1}.ySy.\,y^{-1} \\ &= S \end{aligned}$$

$$\Longrightarrow xy^{-1} \in C_G(S)$$

Hence $C_G(S)$ is a subgroup of G.

Finally, if K be any subset of G with respect to which S is self compressed, then evidently

$$K \subseteq C_G(S)$$

This completes the proof.

Theorem 4.4 - For any subset S of a group G,

$C_G(S) = C_G(G-S)$ where G - S is the compliment of S in G.

Proof. For any $x \in C_G(S)$,

$$S_x = S \Longrightarrow (G - S)_x = G - S \quad \text{(Theorem 2.6)}$$

$$\Longrightarrow \quad C_G(S) \subseteq C_G(G - S)$$

Similarily, we can see that

$$C_G(G - S) \subseteq C_G(G - (G - S))$$

$$\text{or} \quad C_G(G - S) \subseteq C_G(S)$$

Hence

$$C_G(S) = C_G(G - S)$$

This completes the theorem .

Theorem 4.5 - Let $S = \{g, e\}$ be a subset in a group G, then if $x(\neq e) \in C_G(S)$, either $O(x) = 2$ or $O(g) = 2$.

Proof. Let $x(\neq e) \in C_G(S)$, then

$$x S x = S$$

$$\Longrightarrow x g x = g \text{ or } x g x = e$$

If $x g x = g$, then evidently, $O(x) = 2$.

In other case

$$x g x = e$$

$$\Longrightarrow \quad x e x = g$$

Thus

$$x^2 = g \text{ and also } x^{-2} = g$$

$$\Longrightarrow \quad g^2 = e$$

i.e. $O(g) = 2$

Hence the theorem is complete.

3. Relation Between A Set And Its Compressor:

In this section, we show that if a subgroup ~~H of a group~~ H of a group G be compressor of a subset, then the subset is the union of all distinct cosets of H^* with respect to elements of the subset; on the other hand, if a subset be a union of cosets of a subgroup H_1^* of G^*, then the compressor of the subset is the maximal subgroup of G whose smallest c.s.c-subgroup is H_1^*, provided that to every element of compressor, there exists at least one coset of H_1^* in the representation of the subset which is self compressed with respect to that element.

Theorem 4.6-(Correspondence theorem) - If for an arbitrary subset S of a group G, $C_G(S) = H$, then $S = \cup s_i H^*$ where s_i's are all distinct coset representatives of H^* in S. Conversely, if for a subgroup K^* of G^*, we have $S = \cup s_i K^*$ for s_i's in S then H', the largest subgroup for which $H'^* = K^*$ coincides with $C_G(S)$, provided for every x in $C_G(S)$, $(s_i K^*)_x = s_i K^*$ for some $s_i K^*$.

Proof. Let $\quad C_G(S) = H$

$$\Longrightarrow \quad hSh = S \qquad \forall \; h \in H$$

$$\Longrightarrow \quad sh^2 \in S \qquad \forall \; s \in S,\ h \in H$$

$$\Longrightarrow \quad sH^* \subseteq S \qquad \forall \; s \in S$$

$$\Longrightarrow \quad S = \bigcup_{s \in S} sH^* = \cup s_i H^*$$ where s_i's are all distinct coset representatives in S.

Conversely, let $S = \bigcup_{s_i \in S} s_i K^*$. We have for any $h' \in H'$

$$(s_i K^*)_{h'} = h'(s_i K^*)h'$$

$$= s_i(h'^2 K^*)$$

$$= s_i K^* \quad \text{since } h'^2 \in H'^* = K^*$$

$$\Longrightarrow \quad (S)_{h'} = \Big(\bigcup_{s_i \in S} s_i K^*\Big)_{h'}$$

$$= \bigcup_{s_i \in S} s_i K^* = S \qquad \text{(Theorem 2.5)}$$

$$\Longrightarrow \quad H' \subseteq C_G(S)$$

Also, if $x \in C_G(S)$ be arbitrary , then by hypothesis

$$x(s_i K^*)x = s_i K^* \text{ for some } s_i K^*$$

$$\Longrightarrow \quad x^2 \in K^*$$

Thus since H' is the largest subgroup for which $H'^* = K^*$, we have

$$x \in H'$$

$$\Longrightarrow C_G(S) \subseteq H'$$

Hence

$$C_G(S) = H'$$

This proves the theorem.

Cor.4.1 - For a subset S and a subgroup H of a group G, $(C_G(S))^* = H^*$ implies $H \subseteq C_G(S)$.

(Proof follows immediately from the fact that $S = \cup s_i H^*$ for s_i's in S).

Cor.4.2 - For any two subsets S_1, S_2 of a group G, $C_G(S_1) = C_G(S_2) = H$ (say) implies $S_1 \cap S_2 = \cup s_i H^*$ where s_i's are only those elements of S_1 or S_2 for which there exist s_j's $\in S_2$ or S_1 respectively such that $s_j \in s_i H^*$ and $S_1 \cup S_2 = \cup s_i H^*$ where s_i's are distinct representatives of cosets of H^* in $S_1 \cup S_2$.

The following example illustrates and explains the imposition of the additional condition in the converse statement of the theorem.

Example. Let $G = [a]$ be the cyclic group of order 8. Evidently e is a subgroup of G^* and the largest subgroup H of G for which $H^* = e$, is $H = \{a^4, e\}$. Take $S = \{a, a^5\}$ a subset in G. We have

$$S = \{aH^* \cup a^5 H^*\}$$

$$\text{But,} \quad C_G(S) = \{a^2, a^4, a^6, e\} \neq H$$

Hence our condition is necessary in general.

4. Properties Of Compressors Of Subgroups:

In this section, we derive several properties of compressors of subgroups. We first determine a condition, under which the compressor of any subgroup of a group coincides with the subgroup itself, and further show that the compressor of a subgroup contains the compressor of any subgroup of the subgroup. We observe that the compressor of a subgroup whose index in its group is 2, coincides with the whole group but not conversely. It is interesting to note that if for any two finite subgroups containing no element of order 2 in a group, the compressor of one contains the other subgroup, the subgroups must coincide. Finally, we establish that the compressor of any subgroup of a group is the set theoritical union of all elements of the subgroup of elements of order 2 in the factor group of the group by the subgroup.

Def. 4.2 - Let H be a subgroup of a group G. We call c-power of an element $x \in G$ in H to be the least +ve integer n, for which $x^{2^n} \in H$. The c-power of x is taken to be $-\infty$ in H if there exists no such +ve integer n.

It is immediate that c-power of any $x \in C_G(H)$ is 1 in H and conversely. If c-power of an element x in G be n in H, then $x^{2^{n-1}} \in C_G(H)$.

Theorem 4.7 - For any subgroup H of a group G, $C_G(H) = H$ if and only if c-power of all $g \in G - H$ be $-\infty$ in H.

Proof. If $C_G(H) = H$, then there exists no $g \in G$, $g \notin H$ for which $g^2 \in H$, since otherwise

$$gHg = g^2H$$
$$= H$$

$$\Longrightarrow g \in C_G(H)$$

A contradiction that $C_G(H) = H$.

Thus

$$g \in G - H \Longrightarrow g^2 \notin H.$$

Hence, by repeating the argument on $g^2 \in G - H$ etc. we get, ingeneral,

$$g \in G - H \Longrightarrow g^{2^i} \notin H \text{ for any integer } i > 0$$

$$\Longrightarrow \text{c-power of all } g \in G - H \text{ is } -\infty \text{ in } H.$$

Conversely, if c-power of all $g \in G - H$ be $-\infty$ in H, then there exists no element $g \in G - H$, such that $g \in C_G(H)$ since

$$g \in C_G(H) \Longrightarrow g^2 \in H$$

Hence,

$$C_G(H) \subseteq H$$

$$\Longrightarrow \quad C_G(H) = H \quad \text{since } H \subseteq C_G(H)$$

This proves the theorem.

Cor. 4.3- For any subgroup H of a group G, $H \subset C_G(H)$ if and only if c-power of some $g \in G - H$ be other than $-\infty$.

Theorem 4.8 - If for two subgroups H_1, H_2 of a group G, $H_1 \supseteq H_2$ then $C_G(H_1) \supseteq C_G(H_2)$ but not conversely.

Proof. Let $H_1 \supseteq H_2$

$$\Longrightarrow \quad H_1 = \cup\, h_i^{(1)} H_2 \quad \text{where } h_i^{(1)}\text{'s} \in H_1$$

For any $x \in C_G(H_2)$,

$$\begin{aligned} xH_1x &= x(\cup\, h_i^{(1)} H_2)x \\ &= \cup\, x(h_i^{(1)}H_2)x \qquad \text{(Theorem 1.2, (ii))} \\ &= \cup\, h_i^{(1)} H_2 \\ &= H_1 \end{aligned}$$

$$\Longrightarrow \quad C_G(H_1) \supseteq C_G(H_2)$$

To prove the falsity of converse, let

$G = [a]$, the cyclic group of order 12.

$H_1 = [a^2]$, $H_2 = [a^3]$

Then, we have

$$C_G(H_1) = G, \quad C_G(H_2) = H_2$$

Thus here, $C_G(H_1) \supset C_G(H_2)$ but $H_1 \not\supseteq H_2$

This completes the proof.

Theorem 4.9 - If for a subgroup H of a group G, $[G : H] = 2$, then $C_G(H) = G$ but not conversely.

Proof. Let $[G : H] = 2$

$\Longrightarrow$ H is a c.s.c-subgroup, by theorem 2.10

$\Longrightarrow$ $C_G(H) = G$

The converse is false.

Consider, $G = \{a, b, ab, e\}$; $a^2 = b^2 = e, ab = ba$

and $H = e$

We find $C_G(H) = G$ but $[G : H] = 4$

This proves the result.

Theorem 4.10 - If H_1, H_2 be any two periodic subgroups of a group G, containing no element of order 2, then $H_1 \subseteq C_G(H_2)$ together with $H_2 \subseteq C_G(H_1)$ implies $H_1 = H_2$.

Proof. Let $h_1 \in H$, then

$$h_1 e h_1 = h_1^2 \in H_2 \quad \text{since} \quad h_1 \in C_G(H_2)$$

$\Longrightarrow$ $H_1^* \subseteq H_2$

But, evidently, $H_1^* = H_1$

$$\Longrightarrow \quad H_1 \subseteq H_2$$

Similarily, we can see that

$$H_2 \subseteq H_1$$

Consequently

$$H_1 = H_2$$

Hence the proof is complete.

Theorem 4.11 - For any subgroup H of a group G, $C_G(H)$ equals the set theoretical union of all cosets in $\bar{O}_2$, the subgroup of all elements of order 2 in G/H.

Proof. Let a coset $gH \in \bar{O}_2$, then

$$\begin{aligned} gHg &= g^2 H \\ &= (gH)^2 \\ &= H \qquad \text{since } gH \in O_2 \end{aligned}$$

$$\Longrightarrow \quad g \in C_G(H)$$

$$\Longrightarrow \quad gH \subseteq C_G(H) \qquad \text{sinee } H \subseteq C_G(H)$$

$$\Longrightarrow \quad \bigcup_{gH \in \bar{O}_2} gH \subseteq C_G(H)$$

On the contrary, let $x \in C_G(H)$, then

$$xHx = H$$

$$\Longrightarrow \; x^2H = H$$

$$\Longrightarrow \; (xH)^2 = H$$

$$\Longrightarrow \; xH \in \bar{O}_2$$

$$\Longrightarrow \; C_G(H) \subseteq \bigcup_{gH \in \bar{O}_2} gH$$

Hence

$$C_G(H) = \bigcup_{gH \in \bar{O}_2} gH$$

Thus the proof is complete.

Cor. 4.4 - For any two subgroups H_1, H_2 of a group G, $C_G(H_1) = C_G(H_2)$ if and only if the set theoretical unions of cosets in subgroups of all elements of order 2 in G/H_1 and G/H_2 is the same.

5. Calculus Of Compressors:

In the following, we determine the nature of intersection, product and complimentation of compressors of subsets in a group.

Theorem 4.12 - If S_1, S_2 be any two subsets in a group G, then $C_G(S_1) \cap C_G(S_2)$ will be contained in both $C_G(S_1 \cap S_2)$ and $C_G(S_1 \cup S_2)$.

Proof. Let $x \in C_G(S_1) \cap C_G(S_2)$

$$\Longrightarrow \; x \in C_G(S_1), \; x \in C_G(S_2)$$

$$\implies x S_1 x = S_1, \quad x S_2 x = S_2$$

$$\implies x(S_1 \cap S_2)x = S_1 \cap S_2, \ x(S_1 \cup S_2)x = S_1 \cup S_2$$

(Theorem 2.5)

$$\implies C_G(S_1) \cap C_G(S_2) \subseteq C_G(S_1 \cap S_2) \text{ and also } C_G(S_1 \cup S_2)$$

This completes the proof.

Note : The above theorem also holds for an arbitrary family of subsets.

Cor. 4.5 - For any two subgroups H_1, H_2 of a group G,

$$C_G(H_1) \cap C_G(H_2) = C_G(H_1 \cap H_2)$$

(Proof is immediate from the above theorem and theorem 4.8)

Cor. 4.6 - Given two subsets S_1, S_2 of a group G,

(i) If $C_G(S_1 \cap S_2) = O_2$, the subgroup of all elements of order 2 in G, then

$$C_G(S_1) \cap C_G(S_2) = C_G(S_1 \cap S_2)$$

(ii) If $C_G(S_1 \cup S_2) = O_2$, then $C_G(S_1) \cap C_G(S_2) = C_G(S_1 \cup S_2)$.

Theorem 4.13 - Let S_1, S_2 be any two subsets of a group G, then

$$C_G(S_1) \circ C_G(S_2) = C_G(S_1 \cdot S_2)$$

Proof. For an arbitrary element $x \in C_G(S_1)$,

$$x(S_1 \ S_2)x = (x S_1 x) \ S_2$$

$$= S_1 S_2$$

$$\Longrightarrow \quad C_G(S_1) \subseteq C_G(S_1 S_2)$$

Similarily

$$C_G(S_2) \subseteq C_G(S_1 S_2)$$

Hence

$$C_G(S_1) \cdot C_G(S_2) \subseteq C_G(S_1 S_2)$$

This proves the theorem.

Note : If in the above theorem $C_G(S_1 S_2) = O_2$, the subgroup of all elements of order 2 in G, then $C_G(S_1) \cdot C_G(S_2) = C_G(S_1 S_2)$.

Theorem 4.14 - For any subset S of a group G, $G - C_G(S) = G - C_G(G-S)$.

(Proof follows from theorem 4.4)

6. Compressor of A Set And Of Its C-Transform :

Here, we note that the compressors of a subset and of any of its c-transforms are the same in a group; however, if the compressors of any two subsets in a group be the same, one need not be c-transform of other.

Theorem 4.15 - For any subset S of a group G and any $x \in G$, $C_G(S) = C_G(S_x)$. Conversely, if for two subsets S_1, S_2 of G, $C_G(S_1) = C_G(S_2)$, then S_1 is not a C-Transform of S_2 in general.

Proof. One can easily verify that

$$C_G(S_x) = C_G(S)$$

To show that the converse is not true,

Consider $G = [a]$, a cyclic group of order 8

If $S_1 = \{a\}$, $S_2 = \{a^4, a^5\}$

$$C_G(S_1) = C_G(S_2) = \{a^4, e\}$$

But evidently,

$$S_1 \stackrel{C}{\not\longrightarrow} S_1$$

This completes the theorem.

Cor. 4.7 - Let $\{S_\alpha\}_{\alpha \in A}$ *be any family of subsets in a group G,* then for any $x \in G$

(i) $C_G(\bigcap_{\alpha \in A} S_\alpha) = C_G(\bigcap_{\alpha \in A} (S_\alpha)_x)$; (ii) $C_G(\bigcup_{\alpha \in A} S_\alpha) = C_G(\bigcup_{\alpha \in A} (S_\alpha)_x)$

(Proof follows immediately from theorems 4.15 and 1.2)

Cor. 4.8 - For any finite family $\{S_\alpha\}_{\alpha \in A}$ of subsets in a group G and elements $\{x_\alpha\}_{\alpha \in A}$ in G, we have

$$C_G(\prod_{\alpha \in A} S_\alpha) = C_G(\prod_{\alpha \in A} (S_\alpha)_{x_\alpha})$$

(Proof follows from theorems 4.15 and 1.3)

7. Fundamental Mappings And Compressors:

We observe that the isomorphic image of the compressor of a subset in a group under an isomorphism of the group coincides with the compressor of the image-subset in the image group; however, this does not hold in general in case of a homomorphism, but it does hold if the subset be a subgroup containing the kernel of the homomorphism.

Theorem 4.16 - Let ϕ be an isomorphism of a group G onto a group G', then for any subset S of G.

$$(C_G(S))\phi = C_{G'}(S\phi)$$

Proof. Evidently, S is self compressed with respect to $C_G(S)$, hence by theorem 2.16, $S\phi$ is self compressed with respect to $(C_G(S)\phi$. Thus

$$(C_G(S))\phi \subseteq C_{G'}(S\phi) \qquad \text{(Theorem 4.3)}$$

Again, since $S\phi$ is self compressed with respect to $C_{G'}(S\phi)$, we have by theorem 2.16 that S is self compressed with respect to $(C_{G'}(S\phi))\phi^{-1}$. Thus by theorem 4.3,

$$(C_{G'}(S\phi))\phi^{-1} \subseteq C_G(S)$$

$$\Longrightarrow C_{G'}(S\phi) \subseteq (C_G(S))\phi$$

Hence

$$(C_G(S))\phi = C_{G'}(S\phi)$$

This proves the theorem .

Remark : If ϕ be a homomorphism of G, then for any set S only the restricted result $(C_G(S))\phi \subseteq C_{G'}(S\phi)$ holds true, but if S be a subgroup H containing kernel of ϕ, the character of theorem (4.16) is maintained as proved in the following theorem.

Theorem 4.17 - Let ϕ be a homomorphism of a group G onto a group G', then for any subgroup H of G containing K, the kernel of ϕ, we have

$$C_{G'}(H\phi) = (C_G(H))\phi$$

Proof. Evidently, by theorem 2.14 and 4.3

$$(C_G(H))\phi \subseteq C_{G'}(H\phi)$$

Conversely, let $x' \in C_{G'}(H\phi)$ be arbitrary, then

$$x'(H\phi)x' = H\phi$$

$$\Longrightarrow (x\phi)(H\phi)(x\phi) = H\phi \text{ for some } x \in G$$

$$\Longrightarrow (xHx)\phi = H\phi$$

$$\Longrightarrow xHx = H \text{ since } H \supseteq K$$

$$\Longrightarrow x \in C_G(H)$$
$$\Longrightarrow x\phi = x' \in (C_G(H))\phi$$
$$\Longrightarrow C_{G'}(H\phi) = (C_G(H)\phi$$

Hence

$$C_{G'}(H\phi) = (C_G(H))\phi$$

This completes the proof.

8. Direct Products And Compressors :

We note that in a direct product, the direct product of the compressors of subsets of direct factors in the corresponding direct factors is the compressor of the product of these subsets in the direct product group.

Theorem 4.18 - Let a group G be a direct product of its subgroups H_i, $i = 1,2, \dots n$. If S_i denotes subsets of H_i for all i, then

$$C_G\left(\prod_{i=1}^{n} S_i\right) = \mathop{\mathrm{X}}_{i=1}^{n} C_{H_i}(S_i)$$

Proof. We shall first prove that

$$C_G\left(\prod_{i=1}^{n} S_i\right) = \prod_{i=1}^{n} C_{H_i}(S_i)$$

Evidently,

$$\prod_{i=1}^{n} C_{H_i}(S_i) \subseteq \prod_{i=1}^{n} C_G(S_i) \subseteq C_G\left(\prod_{i=1}^{n} S_i\right) \text{ (Theorem 4.13)}$$

On the other hand, if $x \in C_G\left(\prod_{i=1}^{n} S_i\right)$ and $s_i \in S_i$, for $i = 1,2,\dots n$,

then

$$x(s_1 . s_2 \dots s_i \dots s_n)x = s_1' . s_2' \dots s_i' \dots s_n' \in \prod_{i=1}^{n} S_i \ , \ s_i' \in S_i \subseteq H_i$$

or for $x = h_1 \, h_2 \dots \, h_i \dots h_n$, $h_i \in H_i$ in $G = \mathop{\mathrm{X}}_{i=1}^{n} H_i$,

$$(h_1 . h_2 \dots h_i \dots h_n)(s_1 . s_2 \dots s_i \dots s_n)(h_1 . h_2 \dots h_i \dots h_n) = s_1' \dots s_i' \dots s_n'$$

$$\Longrightarrow (h_1 s_1 h_1)(h_2 s_2 h_2) \dots (h_i s_i h_i) \dots (h_n s_n h_n) = s_1' . s_2' \dots s_i' \dots s_n'$$

$$\Longrightarrow h_i\, s_i\, h_i = s_i' \in S_i \quad \text{by uniqueness of representation in } G = \mathop{X}_{i=1}^{n} H_i$$

$$\Longrightarrow h_i\, S_i\, h_i \subseteq S_i \quad \text{for all } i$$

Again, since $x^{-1} = h_1^{-1} \cdot h_2^{-1} \ldots h_i^{-1} \ldots h_n^{-1}$ also belongs to $C_G(\prod_{i=1}^{n} S_i)$, we have as above

$$h_i^{-1}\, S_i\, h_i^{-1} \subseteq S_i \quad \text{for all } i$$

$$\Longrightarrow S_i \subseteq h_i\, S_i\, h_i$$

Hence

$$S_i = h_i\, S_i\, h_i$$

$$\Longrightarrow h_i \in C_{H_i}(S_i)$$

$$\Longrightarrow x = h_1 \cdot h_2 \ldots h_i \ldots h_n \in \prod_{i=1}^{n} C_{H_i}(S_i)$$

$$\Longrightarrow C_G(\prod_{i=1}^{n} S_i) \subseteq \prod_{i=1}^{n} C_{H_i}(S_i)$$

Consequently

$$C_G(\prod_{i=1}^{n} S_i) = \prod_{i=1}^{n} C_{H_i}(S_i)$$

Finally, since $C_{H_i}(S_i) \subseteq H_i$ and $H_i \cap H_j = e$ for $i \neq j$, it is evident that

$$C_G(\prod_{i=1}^{n} S_i) = \mathop{X}_{i=1}^{n} C_{H_i}(S_i)$$

This completes the proof.

Cor. 4.9 - Let a group G be a direct product of its subgroups H_i, $i = 1,2, \dots n$, then if H_i' denotes the subgroup of H_i for all i, we have

$$C_G\left(\mathop{X}_{i=1}^{n} H_i'\right) = \mathop{X}_{i=1}^{n} C_{H_i}(H_i')$$

Theorem 4.19 - Let G_i be arbitrary groups and S_i denote subsets of G_i for all $i = 1,2, \dots n$, then

$$\mathop{X}_{i=1}^{n} C_{G_i}(S_i) = C_{\mathop{X}_{i=1}^{n} G_i}\left(\mathop{X}_{i=1}^{n} S_i\right)$$

Proof. Let $(g_i) = (g_1, g_2, \dots, g_i, \dots g_n) \in \mathop{X}_{i=1}^{n} C_{G_i}(S_i)$ be arbitrary, then

$$(g_i)\left(\mathop{X}_{i=1}^{n} S_i\right)(g_i) = \mathop{X}_{i=1}^{n} (g_i S_i g_i)$$

$$= \mathop{X}_{i=1}^{n} S_i \text{ since } g_i \in C_{G_i}(S_i)$$

$$\Longrightarrow (g_i) \in C_{\mathop{X}_{i=1}^{n} G_i}\left(\mathop{X}_{i=1}^{n} S_i\right)$$

$$\Longrightarrow \mathop{X}_{i=1}^{n} C_{G_i}(S_i) \subseteq C_{\mathop{X}_{i=1}^{n} G_i}\left(\mathop{X}_{i=1}^{n} S_i\right)$$

On the other hand, for any element $(g_i') = (g_1', g_2', \dots, g_i', \dots g_n')$ in $C_{\mathop{X}_{i=1}^{n} G_i}\left(\mathop{X}_{i=1}^{n} S_i\right)$,

$$(g_i')\left(\mathop{X}_{i=1}^{n} S_i\right)(g_i') = \mathop{X}_{i=1}^{n} S_i$$

$$\Longrightarrow \mathop{X}_{i=1}^{n} (g_i' \, S_i \, g_i') = \mathop{X}_{i=1}^{n} S_i$$

$$\Longrightarrow g_i' \, S_i \, g_i' = S_i \quad \text{for all } i$$

$$\Longrightarrow g_i' \in C_{G_i}(S_i)$$

$$\Longrightarrow (g_i') \in \mathop{X}_{i=1}^{n} C_{G_i}(S_i)$$

$$\Longrightarrow C_{\mathop{X}\limits_{i=1}^{n} G_i}\left(\mathop{X}_{i=1}^{n} S_i\right) \subseteq \mathop{X}_{i=1}^{n} C_{G_i}(S_i)$$

Hence

$$\mathop{X}_{i=1}^{n} C_{G_i}(S_i) = C_{\mathop{X}\limits_{i=1}^{n} G_i}\left(\mathop{X}_{i=1}^{n} S_i\right)$$

Thus the proof is complete.

Remark. The above theorem also holds true for complete direct products of groups which can be similarily verified.

9. Compressor And Smallest C.S.C-Subgroup :

We find that the compressor of a subgroup contains the compressor of its smallest c.s.c-subgroup, but it is not without interest to note that the smallest c.s.c-subgroup of the compressor of a subgroup is contained in the compressor of its smallest c.s.c-subgroup. Further if the smallest c.s.c-subgroup of the compressor of a subgroup be contained in another subgroup then the compressor of the first is contained in the compressor of the other.

Theorem 4.20 - Let H, H_1 be any two subgroups of a group G, then

(i) $(C_G(H))^* \subseteq C_G(H^*) \subseteq C_G(H)$

(ii) $(C_G(H))^* \subseteq H_1 \Longrightarrow C_G(H) \subseteq C_G(H_1)$.

Proof. (i) Since $H^* \subseteq H$, the 2nd part of (i) is obvious and

$$C_G(H^*) \subseteq C_G(H) \qquad \text{(Theorem 4.8)}$$

Also, for any $x \in C_G(H)$ and $h \in H$,

$$x^2 h^2 x^2 = xhx \cdot xhx$$

$$= h_1 . h_1 \quad \text{where } h_1 \in H$$

$$= h_1^2 \in H^*$$

$$\Longrightarrow x^2 H^* x^2 \subseteq H^*$$

$$\Longrightarrow H^* = x^2 H^* x^2 \text{ for all } x \in C_G(H) \qquad \text{(Theorem 1.9)}$$

$$\Longrightarrow (C_G(H))^* \subseteq C_G(H^*)$$

This proves (i)

For (ii), $(C_G(H))^* \subseteq H_1$

$$\Longrightarrow \text{For any } x \in C_G(H),\ x^2 \in H_1$$

$$\Longrightarrow H_1 = x^2 H_1$$

$$= x H_1 x \quad \text{for all } x \in C_G(H)$$

$$\Longrightarrow C_G(H) \subseteq C_G(H_1)$$

This completes the theorem.

Cor.4.10 - If for two subgroups H_1, H_2 of a group G, $C_G(H_2) \subseteq C_G(H_1)$ then $C_G(H_1) = C_G(H_2)$ if and only if $(C_G(H_1))^* \subseteq H_2$.

(The sufficiency is immediate from theorem 4.20 (ii) and for necessary part we remark that by theorem 4.6,

$$H_2 = \bigcup_{h_i^{(2)} \in H_2} h_i^{(2)} (C_G(H_1))^* \Longrightarrow (C_G(H_1))^* \subseteq H_2 \;)$$

10. Compressor And Anticenter :

We prove that the intersection of a subgroup with the anticenter of the compressor of the subgroup is contained in the compressor of anticenter of the subgroup . We also note that in a cyclic group the compressor of the anticenter of a subgroup is anticenter of the compressor of the subgroup, but not in general.

Theorem 4.21 - Let H be a subgroup of a group G, then

$$H \cap AC(C_G(H)) \subseteq AC(H) \subseteq C_G(AC(H)).$$

Proof. Since $H \subseteq C_G(H)$, we have by theorem 2 in [6],

$$\begin{aligned} AC(C_G(H)) \cap H &\subseteq AC(H) \\ &\subseteq C_G(AC(H)) \text{ since } AC(H) \text{ is a subgroup.} \end{aligned}$$

This establishes the result

Theorem 4.22 - For any subgroup H of a cyclic group G,

$$AC(C_G(H)) = C_G(AC(H))$$ but it does not hold in general.

Proof. Since G is cyclic, the subgroups H and $C_G(H)$ are also cyclic, hence by theorem 2 in [5]

$$\begin{aligned} AC(C_G(H)) &= C_G(H) \\ &= C_G(AC(H)) \end{aligned}$$

To show, that it does not hold in general

Consider $G = \{a,b,c,e\}$; $a^2 = b^2 = c^2 = e$, $ab = ba = c$,
$bc = cb = a$, $ac = ca = b$

Evidently,

$$AC(G) = e$$

Now if we put H = G, then

$$\begin{aligned} AC(C_G(H)) &= AC(G) \\ &= e \end{aligned}$$

But

$$\begin{aligned} C_G(AC(H)) &= C_G(e) \\ &= G \end{aligned}$$

Thus

$$AC(C_G(H)) \neq C_G(AC(H))$$

This completes the proof.

11. Chain Of Compressors :

For any subgroup H of a group G,

$$H \subseteq C_G(H) \subseteq C_G(C_G(H)) \subseteq \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

is an ascending chain of subgroups of G. We write $C_G(H) = C_G^1(H)$, $C_G(C_G^1(H)) = C_G^2(H)$, $C_G(C_G^2(H)) = C_G^3(H)$ and so on in general $C_G^n(H) = C_G(C_G^{n-1}(H))$. Evidently

$$C_G(H) = \{g \mid g \in G,\ g^2 \in H\}$$

$$C_G^2(H) = \{g \mid g \in G,\ g^{2^2} \in H\}$$

$$C_G^3(H) = \{g \mid g \in G,\ g^{2^3} \in H\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$C_G^n(H) = \{g \mid g \in G,\ g^{2^n} \in H\}$$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ and so on. If an element $g \in C_G^n(H)$ and $g \notin C_G^{n-1}(H)$ then $g^{2^n} \in H$, $g^{2^{n-1}} \notin H$. We now prove the following important theorem which gives a criterion for the chain of compressors to be finite.

Theorem 4.23 - Let H be a subgroup of a group G. The chain

$$C_G^1(H) \subseteq C_G^2(H) \subseteq C_G^3(H) \quad \ldots\ldots\ldots \subseteq C_G^n(H) \subseteq \ldots\ldots\ldots \quad (1')$$

breaks off at n^{th} stage $(n > 1)$ if and only if least upper bound of c-powers in H of elements in G - H be n. The chain breaks off at the Ist stage if c-powers in H of all elements in G - H be < 2. (For c-power, sec. Def. 4.2).

Proof. Let, the chain (1') breaks off at the n^{th} $(n > 1)$ stage, then

$$C_G^n(H) = C_G^k(H) \quad \text{for all } k \geqslant n.$$

$\Longrightarrow$ For all $g \in G - H$, either $g \in C_G^n(H)$ or $g \notin C_G^i(H)$ for all $(i \geqslant 1)$

$\Longrightarrow$ Either $g^{2^n} \in H$ or $g^{2^i} \notin H$ for all integers $i \geqslant 1$.

$\Longrightarrow$ C-power of every $g \in G - H$ is $\leqslant n$ since the C-power in the last case is $-\infty$.

Now, as $C_G^n(H) \supset C_G^{n-1}(H)$ there exists a $g' \in G - H$ such that

$$g' \in C_G^n(H), \quad g' \notin C_G^{n-1}(H)$$

or $g' \in C_G^n(H)$, $g' \notin C_G^k(H)$ for $(1 \leqslant k \leqslant n-1)$

$\Longrightarrow$ $g'^{2^n} \in H$, $g'^{2^k} \notin H$ for $(1 \leqslant k \leqslant n-1)$

$\Longrightarrow$ C-power of $g' \in G - H$ is n in H.

Hence, least upper bound of c-powers in H of all elements, in G - H is n.

Conversely, if least upper bound of c-powers in H of all elements in G - H be n, then for all $g \in G - H$, c-power in H is $\leq n$.

$$\Longrightarrow g^{2^i} \in H \text{ for some } 1 \leq i \leq n \text{ or } g^{2^j} \notin H \text{ for all integers } j \geq 1$$

$$\Longrightarrow C_G^n(H) = C_G^k(H) \quad \text{for all } k \geq n$$

Again, due to property of least upper bound, there exists a $g_1 \in G - H$ such that c-power of g_1 be n in H. Thus

$$g_1^{2^n} \in H, \quad g_1^{2^i} \notin H \quad \text{for all } 1 \leq i \leq n-1$$

$$\Longrightarrow \quad g_1 \in C_G^n(H), \quad g_1 \notin C_G^{n-1}(H)$$

$$\Longrightarrow \quad C_G^n(H) \supset C_G^{n-1}(H))$$

$\Longrightarrow$ The chain (1') breaks off at n^{th} stage.

Finally, if c-powers in H of all elements in G - H be < 2, then clearly the chain (1') breaks off at the Ist stage.

This completes the theorem.

Remark : The chain in the above theorem breaks off always at $C_G^n(H) = G$ if c-power of every element in G - H is i such that $1 \leq i \leq n$ otherwise $C_G^n(H)$ is a proper subgroup of G.

12. Results True For Non-Abelian Groups :

We have proved all the results given above for abelian groups only, however, it is easy to verify that the theorems 4.1,4.2,4.4, 4.5,4.12,4.14,4.16,4.17 and 4.19 also hold for non-abelian groups with no change in the proofs already supplied.

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CHAPTER - FIVE

COMPRESSION CLASSES AND POWER C - TRANSFORMS

1. Introduction : In this chapter we deal with compression classes (§2) and power C-transforms (§8). We first prove that, in a group, the cardinal number of elements in any 'class of compressed subsets' containing a subset S is the index of the compressor of S in the group. It is interesting to note that the behaviour of C-transforms with respect to compressor is just analogous to that of conjugate transforms with respect to normalizer. Further, we show that 'Any collection of classes of compressed elements generate a c.s.c-subgroup', and a number of other properties such as 'Product of a class of compressed elements in a group by any element of the group is again a class of compressed elements', 'Set of all elements inverse to elements of a class of compressed elements is also a class of compressed elements', 'The set product of any two classes of compressed elements also comes out to be a class of compressed elements'. Some of our important observations are that if every class of compressed elements in a subgroup of finite index in a group be finite, then every class of compressed elements in the group is also finite, also, 'If a group has finite number of classes of compressed elements, then the index of the subgroup of all elements of order 2 of the group is finite if and only if the group is finite' etc.

We define a group to be locally completly self compressed if its every class of compressed elements be finite, and prove that a group is locally completly self compressed if and only if its every finite subset be contained in a finite c.s.c-subset.

We also define a notion of 'power C-transform' of a subset with respect to a given element in a group, and prove, apart from some other properties, that 'The power C-transform of an element with respect to an element of order n has n or n/2 elements according as n is odd or even', and that 'The power C-transforms of two subsets with respect to a given element are either identical or disjoint'.

We relate the equality of two classes of compressed elements with the equality of two power C-transforms in a certain sense (Theorem 5.13). Finally, we have shown that an element of a group lies in the compressor of a subset of power n in the group only if the order of the element divides 2n .

2. Compression Classes:

From theorem 1.1, the relation of compression is an equivalence relation in a group, hence the family Σ of all subsets of a group can be partitioned into mutually disjoint sub-classes Σ_i called 'classes of compressed subsets' such that whenever $S_{i_1}, S_{i_2} \in \Sigma_i$, we have

$$S_{i_1} \xrightarrow{C} S_{i_2}$$

but if $S_i \in \Sigma_i$, $S_j \in \Sigma_j$ for $(i \neq j)$, then $S_i \overset{C}{\not\longrightarrow} S_j$. Further, since from theorem 1.6, all subsets in one class Σ_i have the same cardinal number, there exists a partition of elements of a group into disjoint subsets called 'classes of compressed elements'. Evidently, any class of compressed subsets containing a fixed subset S_1 (or of elements containing a fixed element g_1) in a group G is just the set of all compressions of S_1 (or g_1) with respect to elements of G, and, hence, is a c.w.c-subset of G. By theorem 1.8, if any C-transform of a subgroup be a subgroup then it coincides with the subgroup itself. Thus no class of compressed subsets contains more than one subgroup.

3. Number of C-Transforms Of A Subset:

We establish that number of C-transforms of a subset in a group is the index of the compressor of the subset in the group.

Theorem 5.1 - For any subset S of a group G, the cardinal number of the set of all C-transforms of S with respect to elements in G is $[G : C_G(S)]$.

Proof. Let $x \in C_G(S)$ and $S_{g_1} = S_1$ for $g_1 \in G$, then

$$S_{(g_1 x)} = S_1$$

Also, if $S_1 = S_{g_2}$ for $g_2 \in G$, then

$$\begin{aligned} S_{(g_2 g_1^{-1})} &= g_2\, g_1^{-1} \,.\, S.\; g_2\, g_1^{-1} \\ &= g_1^{-1}\, g_2\, S\, g_2\, g_1^{-1} \\ &= g_1^{-1}\, S_1\, g_1^{-1} \\ &= g_1^{-1}\, g_1\, S\, g_1\, g_1^{-1} \quad \text{since } Sg_1 = S_1 \\ &= S \end{aligned}$$

$$\implies g_2 g_1^{-1} \in C_G(S)$$

$$\implies g_2 \in g_1\, C_G(S)$$

Thus the elements g_1, g_2 lie in the same left coset of $C_G(S)$. Hence there exists a 1-1 correspondence between the left cosets of $C_G(S)$ in G and C-transforms of S in G.

This completes the proof.

Cor.5.1 - The number of C-transforms of any element g of a group G is $[G : O_2]$ where O_2 denotes the subgroup of all elements of order 2 in G.

(Proof follows immediatly by consideration of the compressor of any element).

Remark : Since the index of a subgroup of a finite group divides the order of group, the number of compressed transforms of any subset of a finite group divides the order of the group.

4. Properties Of Classes Of Compressed Subsets:

In this section, we investigate some basic properties of classes of compressed elements. We find that the product of any element of the group with any class, the set product of any two classes, the collection of inverses of elements in a class form again a class of compressed elements. Any collection of classes of compressed elements generates a c.s.c-subgroup. These properties also hold good for classes of compressed subsets. We note that in a finite group there exists no subset property contained in its compressor which contains elements from all classes of compressed elements in the group.

Theorem 5.2 - For any group G,

(i) If $\{g_\alpha\}_{\alpha \in A}$ be any class of compressed elements in G and $g' \in G$, then $\{g' g_\alpha\}_{\alpha \in A}$ is also a class of compressed elements in G, A being an arbitrary index set.

(ii) The set product $K_1 K_2$ of any two classes of compressed elements K_1, K_2 of G is a class of compressed elements in G.

(iii) If K is a class of compressed elements in G, then so is K^{-1}, the set of all inverses of elements in K.

Proof. (i) For any $g \in G$ and some $g_{\alpha_1} \in \{g_\alpha\}_{\alpha \in A}$,

$$g(g' g_{\alpha_1})g = g'(g g_{\alpha_1} g)$$

$$= g' g_{\alpha_j} \in \{g' g_\alpha\}_{\alpha \in A}, \quad g_{\alpha_j} = g g_{\alpha_1} g \in \{g_\alpha\}_{\alpha \in A}$$

On the other hand, for any $g_{\alpha_i} \in \{g_\alpha\}_{\alpha \in A}$, since $\{g_\alpha\}_{\alpha \in A}$ is a class of compressed elements, there exists an element $g'' \in G$ such that

$$g'' g_{\alpha_1} g'' = g_{\alpha_i}$$

$$\Longrightarrow g''(g' g_{\alpha_1})g'' = g'(g'' g_{\alpha_1} g'')$$

$$= g' g_{\alpha_i}$$

Hence $\{g'g_\alpha\}_{\alpha\in A}$ is a class of compressed elements in G.

This proves (i).

(ii) By (i), for any fixed $k_1' \in K_1$, $k_1' K_2$ is a class of compressed elements in G. Now if we show that for any $k_1 \in K_1$,

$$k_1 K_2 = k_1' K_2$$

We sshall prove that $K_1 K_2$ is a class of compressed elements in G. Since $k_1 K_2$ and $k_1' K_2$ are two equivalence classes with respect to the relation of compression,

$$k_1 K_2 \cap k_1' K_2 = \phi \quad \text{or} \quad k_1 K_2 = k_1' K_2$$

Further since K_1 is a class of compressed elemmnts, there exists an element $g \in G$ such that

$$g k_1' g = k_1$$

$$\Longrightarrow g(k_1 k_2)g = (g k_1' g)\, k_2$$

$$= k_1 k_2 \quad \text{for all } k_2 \in K_2$$

$\Longrightarrow k_1 k_2 \in k_1' K_2$ for all $k_2 \in K_2$, since $k_1' K_2$ is a class of compressed elements in G.

$$\Longrightarrow k_1 K_2 \cap k_1' K_2 \neq \phi$$

$$\Longrightarrow k_1 K_2 = k_1' K_2 \quad \text{for all } k_1 \in K_1$$

This establishes (ii)

(iii) Let $k' \in K$ be a fixed element. For any $g \in G$,

$$g k'^{-1} g = (g^{-1} k' g^{-1})^{-1} \in K^{-1}$$

On the other hand, if $k \in K$ be arbitrary, since K is a class of compressed elements, there exists some $g \in G$ such that

$$g k' g = k$$

$$\Longrightarrow g^{-1} k'^{-1} g^{-1} = k^{-1} \in K^{-1}$$

Hence K^{-1} is a class of compressed elements in G.

This completes the theorem.

Cor.5.2 - If $\{g_\alpha\}_{\alpha \in A}$ be a class of compressed elements in a group G, then for any subset S of G, $\{g_\alpha S\}_{\alpha \in A}$ is a class of compressed subsets in G.

(Proof is trivial)

Theorem 5.3 - Any collection of classes of compressed elements in a group G generates a c.s.c-subgroup of G.

Proof. Let S denotes a collection of classes of compressed elements in G, then since every class of compressed elements is a c.s.c- subset of G, it implies from theorem 2.5 that S is a c.s.c-subset of G which further implies from theorem 2.7 that S generates a c.s.c-subgroup in G. This is what we had to establish.

Note : We can easily verify that theorems 5.2 and 5.3, and Cor.5.2 also hold true in case of classes of compressed subsets in a group.

Remarks: (i) We assert that no subset S of a finite group G such that $C_G(S) \supset S$ can contain elements belonging to every class of compressed elements in G. If there exists such a subset S, the class of compressed subsets containing S will contain all elements of G, whereas the subsets in this class cannot contain elements more than $[G : C_G(S)] \cdot O(S)$. Evidently $[G : C_G(S)] \cdot O(S) < O(G)$ since $O(C_G(S)) > O(S)$. Hence the assertion follows.

(ii) It is evident that any c.s.c-subset of a group is a union of classes of compressed elements.

5. Condition For Finiteness Of A Class Of Compressed Elements:

We find out a necessary and sufficient condition for any class of compressed elements in a group to be finite and establish that if every class of compressed elements in a subgroup of finite index in its group be finite, then every class of compressed elements in the group is also finite. In the following theorem we establish the finiteness condition:

Theorem 5.4- An element g_1 of a group G lies in a finite class of compressed elements if and only if there exists two subsets S_1, S_2 of G such that $O(S_1 \cap S_2) = 1$ and $[G : C_G(S_i)] < \infty$ for $i = 1,2$.

Proof. Let g_1 lies in a finite class of compressed elements, then since

$C_G(g_1) = O_2$, the subgroup of all elements of order 2 in G.

$$\Longrightarrow [G : O_2] < \infty \qquad \text{(Theorem 5.1)}$$

Now, for any subset S of G, since $O_2 \subseteq C_G(S)$

$$[G : C_G(S)] < \infty$$

Thus the condition is obviously necessary.

Conversely, since

$$C_G(S_1) \cap C_G(S_2) \subseteq C_G(S_1 \cap S_2) \qquad \text{(Theorem 4.12)}$$

$$\Longrightarrow \quad C_G(S_1) \cap C_G(S_2) \subseteq O_2 \qquad \text{since } O(S_1 \cap S_2) = 1$$

But, evidently

$$O_2 \subseteq C_G(S_i) \quad \text{for} \quad i = 1,2$$

$$\Longrightarrow \quad C_G(S_1) \cap C_G(S_2) = O_2$$

Now $[G : C_G(S_i)] < \infty$ for $i = 1,2$ and $[G : C_G(S_1) \cap C_G(S_2)] \leq \prod_{i=1}^{2} [G : C_G(S_i)]$

which can be easily verified by theorems 1.5.3 and 1.5.5 [5]

$$\Longrightarrow [G : O_2] \leq \prod_{i=1}^{2} [G : C_G(S_i)] < \infty$$

$\Longrightarrow$ Every class of compressed elements in G finite.

This completes the theorem.

Cor.5.3 - Any element of a group G lies in a finite class of compressed elements if and only if every subset of the group lies in a finite class of compressed subsets.

Theorem 5.5 - Let H be a subgroup of finite index in a group G, then if every class of compressed elements in H be finite implies every class of compressed elements in G is finite.

Proof. If $[G : H] = n$, we have $G = \bigcup_{i=1}^{n} g_i H$ where $g_i H \cap g_j H = \phi$ for $i \neq j$.

Since every class of compressed elements in H is finite, we have by cor 5.1,

$[H : O_2'] < \infty$ where O_2' is the subgroup of all elements of order 2 in H.

$\Longrightarrow [G : O_2'] < \infty$ since $[G : H] = n$

$\Longrightarrow [G : O_2] < \infty$ since $O_2' \subseteq O_2$, the subgroup of all elements of order 2 in G.

$\Longrightarrow$ Every class of compressed elements in G is finite. (Cor.5.1)

This completes the proof.

6. Groups Having Finite Number Of Classes Of Compressed Elements:

We investigate below two criteria fro a group to have *finite number of classes of compressed elements.*

Lemma 5.1 - The class of compressed elements K_{g_1} in a group G containing an element g_1 of G is $g_1 G^*$.

Proof, Since for any $g \in G$,

$$g g_1 g = g_1 g^2$$

$$\Longrightarrow K_{g_1} = g_1 G^*$$

This proves the lemma.

Note : The lemma simplifies the proof of theorem 5.2 completely.

Theorem 5.6 - A group G has finite number of classes of compressed elements if and only if $[G : G^*]$ is finite, and the number is exactly the index of G^* in G.

Proof. (It is immediate from lemma 5.1)

Theorem 5.7 - Let O_2 denotes the subgroup of all elements of order 2 in a group G, then if G has finite number of classes of compressed elements, $[G : O_2] < \infty$ if and only if G is finite.

Proof. If G be finite, the proof is trivial. Conversely, let

$$[G : O_2] < \infty$$

==> Every class of compressed elements in G is finite (Cor.5.1)

==> G is finite.

This proves the theorem.

Cor.5.4 - An infinite group G has finite number of classes of compressed elements only if $[G : O_2] \nless \infty$.

Remark : The above results have been established for arbitrary groups. In case of finite groups, it can be readily seen that the number of classes of compressed elements in a group is equal to the number of all elements in G whose order is ≤ 2.

Appealing to theorem 5.6 and Cor. 3.7, we deduce that the number of classes of compressed elements in a finitely generated groups is also finite and equals to $2^{r_0(G)+r_2(G)}$. In general, if $r_0(G) + r_2(G) < \infty$ in a group having basis, we assert the same as for finitely generated groups. Thus we observe that, except for infinitely generated groups without a basis, as long as $r_0(G)$ and $r_2(G)$ are finite, the number of all classes of compressed elements of the group will always be finite and a power of 2.

7. Locally C.S.C-Groups :

Def. 5.1 - We call G a locally c.s.c-group if every class of compressed elements in G be finite.

Examples: (i) Every finite group.

(ii) Any group G for which $[G : O_2] < \infty$ where O_2 denotes the subgroup of all elements of order 2 in G, for instance periodic groups having a finite number of elements of order other than 2.

We find a necessary and sufficient condition for a group to be locally completly self compressed. The requirements is that its every finite subset should lie in a finite c.s.c-subset. We also establish that in a group, having finite layer of elements of order 2, finiteness is necessary and sufficient for the group to be locally completly self compressed.

Theorem 5.8 - A group G is locally completly self compressed if and only if every finite subset of G is contained in a finite c.s.c-subset of G.

Proof. Let G be a locally c.s.c-group. Consider

$$S = \{g_1, g_2, \dots, g_n\}$$

a finite subset in G. If K_{g_i}, $i = 1,2, \dots n$ denotes the class of compressed elements in G containing the element g_i, then since K_{g_i} is a c.s.c-subset of G,

$$S' = \bigcup_{i=1}^{n} K_{g_i} \supseteq S$$

is, by theorem 2.5, a c.s.c-subset of G, and is finite since each K_{g_i} is finite. Hence the condition is necessary.

Conversely, if the condition be satisfied, then since for any element $g \in G$, any c.s.c-subset of G containing g, contains the class K_g of compressed elements containing g, it follows

$$O(K_g) < \infty$$

This proves the theorem.

Cor.5.5 - A periodic group G is locally completly self compressed if and only if every finite subset of G is contained in a finite c.s.c-subgroup of G.

(Proof follows immediately from theorems 5.8 and 2.7)

Theorem 5.9 - A group G with finite layer of elements of order 2 is locally completly self compressed if and only if G is finite.

Proof. If G be finite the result is obvious . Conversely, since G is locally c. s. c-group every class of compressed elements in G is finite, hence by Cor. 5.1

$[G : O_2] < \infty$ where O_2 denotes the subgroup of all elements of order 2 in G.

$\Longrightarrow$ $O(G) < \infty$ since $|O_2| < \infty$

Hence the theorem is proved.

8. A Power C-Transform And Its order:

We in this section, introduce the concept of power C-transform of a subset in a group with respect to an element of the group, and discuss the variation of its order in different circumstances.

Def. 5.2 - Let S be a subset of a group G, the set

$$S^{(g)} = \{g^i \, S \, g^i \mid i = 0, \pm 1, \pm 2, \ldots\} \text{ for } g \in G$$

is called the 'power C-transform' of S with respect to g.

Evidently if $O(g) = \infty$, for any element $g_1 \in G$, $O(g_1^{(g)}) = \infty$; however, if $O(g) < \infty$, then we have the following theorem :

Theorem 5.10 - Let g, g_1 be any two elements of a group G. If $O(g) = n$, we have $O(g_1^{(g)}) = n$ or $n/2$ according as n is odd or even.

Proof. Evidently

$$g_1^{(g)} = \left\{ g^i g_1 g^i \right\} \quad i = 1, 2, \ldots n \Big\}$$

Let $1 \leq (n_1 > n_2) \leq n$, then if

$$g^{n_1} g_1 \, g^{n_1} = g^{n_2} \, g_1 \, g^{n_2}$$

$$\Longrightarrow \quad g_1 \, g^{2n_1} = g_1 \, g^{2n_2}$$

$$\Longrightarrow \quad g^{2(n_1 - n_2)} = e$$

$$\Longrightarrow \quad n \mid 2(n_1 - n_2)$$

But, since $0 < 2(n_1 - n_2) < 2n$, we have

$$2(n_1 - n_2) = n$$

A contradiction in case n be odd, hence if n be odd

$$g^{n_1} g g^{n_1} \neq g^{n_2} g g^{n_2} \quad \text{for all} \quad 1 \leq n_1, n_2 \leq n, \; n_1 \neq n_2$$

$$\Longrightarrow \quad O(g_1^{(g)}) = n$$

Again, if n be even

$$n_1 = n/2 + n_2$$

On the other hand, if for $1 \leq (n_1 > n_2) \leq n$, we have $n_1 = n/2 + n_2$, then

$$g^{n_1} g_1 g^{n_1} = g^{n/2+n_2} g_1 g^{n/2+n_2}$$

$$= g^{n_2} g_1 g^{n_2} g^n$$

$$= g^{n_2} g_1 g^{n_2}$$

$$\Longrightarrow O(g_1^{(g)}) = n/2$$

This completes the theorem.

Remark : If $O(g) = \infty$, for any subset S of G, $O(S^{(g)}) \neq \infty$ in general and also the above theorem need not hold. For example, let

$G = [a]$, infinite cyclic group generated by a.

and

$S = G$

Then, for any $g \in [a]$

$$g^{i} S g^{i} = S \text{ for all } i \in I$$

$$\Longrightarrow O(S^{(g)}) = 1 \text{ for all } g \in G$$

Cor.5.6 - Let g, g_1 be any two elements of a group G such that $O(g) < \infty$, then

(i) $O(g_1^{(g)}) = O([g^2])$ (ii) $O(g_1^{(g)}) = O(g_1^{(g^i)})$ where $(i, O(g))=1$

(Proof is trivial in view of theorem 5.10)

Cor. 5.7 - In a group G, for g, g_1, g', g_1' in G, if $O(g) \geq O(g')$, then

$$O(g_1^{(g)}) = O(g_1'^{(g')}) \text{ implies } O(g) = O(g') \text{ or } 2O(g')$$

Theorem 5.11 - Let G be a group and $g \in G$ such that $O(g) = m$; if $d > 0$ such that $d|m$, then for any $g_1 \in G$, we have

(i) $O(g_1^{(g)}) = d\, O(g_1^{(g^d)})$ if m is odd.

In case m be even, we have

(ii) $O(g_1^{(g)}) = d\, O(g_1^{(g^d)})$ or $\frac{1}{2}\, d\, O(g_1^{(g^d)})$ according as $(2, m/d) = 2$ or $(2, m/d) = 1$.

Proof. (i) Let m be odd, then from Cor. 5.6,

$$\begin{aligned} O(g_1^{(g)}) &= O([g^2]) \\ &= O([g]) && \text{since } (2, m) = 1 \\ &= d\, O([g^d]) && \text{as } (d, m) = d \\ &= d\, O([g^{2d}]) && \text{since } (2, m/d) = 1 \\ &= d\, O(g_1^{(g^d)}) && \text{(cor 5.6)} \end{aligned}$$

(ii) Let m be even then again from Cor. 5.6,

$$\begin{aligned} O(g_1^{(g)}) &= O([g^2]) \\ &= \frac{1}{2} O([g]) && (2, m) = 2 \\ &= \frac{d}{2} O([g^d]) . && \text{since } (d, m) = d \end{aligned}$$

Now, if $(2, m/d) = 1$, we have

$$O(g_1^{(g)}) = \frac{d}{2} O([g^{2d}])$$

$$= \frac{d}{2}\, O(g_1^{(g^d)})$$

and if $(2, m/d) = 2$,

$$O(g_1^{(g)}) = \frac{d}{2} . 2\, O([g^{2d}])$$
$$= d\, O(g_1^{(g^d)})$$

This completes the theorem.

We shall now prove a theorem regarding the order of a power C-transform of a subset in a group. To prove this we define:

Def. 5.3 - Let S be a subset of a group G and $g \in G$, then the smallest +ive integer n for which $S_{g^n} = S$ is called the 'compression exponent' of g with respect to S in symbols $C_E(g,S) = n$. If there exists no such +ive integer n, then we write $C_E(g,S) = \infty$.

The following properties of compression exponent are interesting.

(i) $S_g = S$ if, and only if, $C_E(g,S) = 1$

(ii) $C_E(g,S) = C_E(g^i,S)$ for every $(i, C_E(g,S)) = 1$, $i > 0$.

Theorem 5.12 - Let S be a subset of a group G and $g \in G$, then $O(S^{(g)}) = n$ if, and only if, $C_E(g,S) = n$.

Proof. Let $C_E(g,S) = n$, then if for $0 \leq (n_1 > n_2) < n$, we have

$$S_g^{n_1} = S_g^{n_2}$$

$$\Longrightarrow S_g^{n_1 - n_2} = S \quad \text{where} \quad 1 \leq n_1 - n_2 < n$$

A contradiction that $C_E(g,S) = n$, hence

$$O(S^{(g)}) \geq n$$

Again, if m be any integer $> n$, then there exist integers r and s such that

$$m = rn + s \qquad \text{where} \quad 0 \leq s < n$$

$$\Longrightarrow g^m S g^m = g^{rn+s} S g^{rn+s}$$

$$= g^s S g^s$$

$$\Longrightarrow O(S^{(g)}) = n$$

Conversely, the result follows from contradiction.

Hence the theorem is complete.

9. Partition Of Subsets In Relation To Power C-Transforms:

We find that any two distinct subsets of a group have either identical or disjoint power C-transforms with respect to any given element of the group. This suggests a partition of the family of all subsets of the group with respect to any given element of the group. The partition is, in general, different with respect to different elements. We also observe that no two distinct subgroups have identical power C-transforms with respect to any element of the group.

Theorem 5.13 - For any two distinct subsets S_1, S_2 of a group G and $g \in G$, $S_1^{(g)} \cap S_2^{(g)} = \phi$ or $S_1^{(g)} = S_2^{(g)}$.

Proof. Let

$$S_1^{(g)} \cap S_2^{(g)} \neq \phi$$

$$\Longrightarrow \quad g^r S_1 g^r = g^s S_2 g^s \quad \text{for some integers } r \text{ and } s.$$

Here $r \neq s$ and also $O(g) \nmid (r-s)$, since otherwise, we get $S_1 = S_2$, a contradiction to supposition that $S_1 \neq S_2$.

Now, we have

$$S_1 = g^{s-r}\, S_2\, g^{s-r}$$

$$\Longrightarrow \quad g^i S_1 g^i \in S_2^{(g)} \quad \text{for all integers } i$$

$$\Longrightarrow \quad S_1^{(g)} \subseteq S_2^{(g)}$$

Also, since

$$S_2 = g^{r-s}\, S_1\, g^{r-s}$$

$$\Longrightarrow \quad S_2^{(g)} \subseteq S_1^{(g)}$$

Consequently

$$S_1^{(g)} = S_2^{(g)}$$

This proves the theorem.

Cor. 5.8 - For any two subsets S_1, S_2 of a group G and $g \in G$, $S_1^{(g)} = S_2^{(g)}$ if, and only if S_1 is a C-transform of S_2 with respect to an element of the cyclic group $[g]$.

(Proof is immediate from theorem 5.12)

Cor. 5.9 - Let H_1, H_2 be any two distinct subgroups of a group G and $g \in G$ then $H_1^{(g)} \neq H_2^{(g)}$.

(Proof of the corollary follows immediatly from theorem 1.8).

Remark : From theorem 5.12, any family Σ of subsets of a group G can be partitioned into disjoint subclasses Σ_i with respect to any element $g \in G$ in such a way that power C-transforms with respect to g of all sets in one Σ_i are the same but whenever $S_i \in \Sigma_i$, $S_j \in \Sigma_j$ $(i \neq j)$ then

$$S_i^{(g)} \cap S_j^{(g)} = \phi$$

Further, if for any two subsets S_1, S_2 of G and $g \in G$, $S_1^{(g)} = S_2^{(g)}$ then by theorem 1.6, S_1 and S_2 have same cardinal number but the converse is not true. For example, let

$G = [a]$, the infinite cyclic group generated by a.

$$S_1 = a, \quad S_2 = a^2, \quad g = a$$

Then, evidently

$$S_1^{(g)} \neq S_2^{(g)} \quad \text{since } e \in S_2^{(g)} \text{ but } \notin S_1^{(g)}$$

Thus

$$O(S_1) = O(S_2) \not\Rightarrow S_1^{(g)} = S_2^{(g)}$$

Also, it is evident that the partition of Σ with respect to different elements $g \in G$ is, in general, different. In a cyclic group $G = [a]$, the class of compressed subsets containing a given subset coincides with the power C-transform of the subset with respect to a.

10. Classes Of Compressed Elements And Power C-Transforms:

The following theorem gives a relationship between power C-transforms of elements and the classes of compressed elements.

Theorem 5.14 - Let g_1, g_2 be any two elements of a group G and K_{g_i}, $i = 1,2$ denote the classes of compressed elements containing g_i then $K_{g_1} = K_{g_2}$ if and only if $g_1^{(g)} = g_2^{(g)}$ for some $g \in G$.

Proof. Let

$$K_{g_1} = K_{g_2}$$

$$\Longrightarrow \quad g_1 \in K_{g_2}$$

$$\Longrightarrow \quad g_1 = gg_2g \quad \text{for some } g \in G$$

$$\Longrightarrow \quad g_1^{(g)} = g_2^{(g)} \quad (\text{cor } 5.8)$$

Conversely, if

$$g_1^{(g)} = g_2^{(g)}$$

$$\Longrightarrow \quad g_1 = g^i g_2 g^i \quad \text{for some integer } i$$

$$\Longrightarrow \quad g'g_1g' = g'g^i.\ g_2.g^ig' \quad \text{for all } g' \in G$$

$$= g'g^i.\ g_2.g'g^i \in K_{g_2}$$

$$\Longrightarrow \quad K_{g_1} \subseteq K_{g_2}$$

Similarily, we can see that

$$K_{g_2} \subseteq K_{g_1}$$

Hence

$$K_{g_1} = K_{g_2}$$

This completes the proof.

Remark : The above theorem also holds true for subsets. One may easily verify that the relation that 'any two subsets which have same power C-transforms with respect to some element of G are related' is an equivalence relation. Further, it is immediate, in view of theorem 5.14, that the partition of the family of all subsets Σ in G with respect to this relation is in fact the partition of Σ into classes of compressed subsets.

11. Power C-Transform And Compressor:

The results in this section enlighten the structure of compressor of finite subset in a group.

Theorem 5.15- Let for a subset S of a group G, $O(S) = n$, then $x \in C_G(S)$ implies $O(x) | 2n$.

Proof. $x \in C_G(S)$

$\implies S = x\,S\,x$

$\implies S = x^i\,S\,x^i$ for all integers i

$\implies S = S^{(x)}$

$\implies s^{(x)} \subseteq S$ for all $s \in S$

$\implies O(x) < \infty$

Also, from theorems 5.10 and 5.13, for some natural numbers k and k_1,

$$O(S) = n = O(x)k_1 \text{ or } O(x)/2\cdot k \text{ according as } O(x) \text{ is odd or even.}$$

$$\implies O(x) \mid 2n$$

This completes the proof.

Cor.5.10 - For any subset S of a group G such that $O(S) = n$, if $x \in C_G(S)$ has order greater than n, then $O(x) = 2n$.

(Proof is trivial)

Cor. 5.11 - Let S be a subset of a group G and $x \in C_G(S)$, then

$$O(S) > O(x)/2$$

(Proof is immediate)

Cor. 5.12 - Let G be a finite group and $[G : O_2] = r$, where O_2 denotes the subgroup of all elements of order 2 in G then for any $g \in G$, $O(g) \mid 2r$.

(Proof of the corollary follows from theorem 5.15 and from the fact that each class of compressed elements in G has r elements and it is a c.s.c-subset of G) .

12. Theorems True in Non-Abelian Groups:

This chapter is devoted only to the study of abelian groups but it is immediate that the theorems 5.12, 5.13 and cor.5.8 hold also for non-abelian groups without any substantial change in arguments of the proofs already supplied.

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CHAPTER - SIX

GENERALISED - COMPRESSOR

1. Introduction: This chapter studies a generalisation of the notion of compressor in a group in view of the notion of 'compressed transform' given in chapter one. For a subset S of a group G, we have defined $C_G(S)$ to be the set of all elements $x \in G$ such that $S_x = S$ (defn 4.1) but if S_1, S_2 be any two distinct subsets of G, the set of all elements $x \in G$ such that $(S_1)_x = S_2$ we call 'S_1-compressor-S_2' or 'generalised compressor of (S_1, S_2)' in G (defn 6.1). We show that if the generalised compressor of a pair of subsets in a group be non-null, it coincides with a coset of the compressor of the first subset, however, if the orders of two given subsets of a group be distinct, the generalised compressor of the pair of these subsets (in whatever order) is null, the converse of the latter is not true in general. The criterion for a group G to have a pair of elements with null generalised compressor, as we investigate, comes out to be that the group should contain its smallest c.s.c-subgroup properly. Besides finding out some basis properties connected with intersection, product etc. of generalised compressors, we have shown that an isomorphic image of the generalised compressor of a pair of subsets in a group is the generalised compressor of the pair of image subsets in the same order in the image group, and also note that the result holds in a restricted sense in case of a homomorphism, however, this restriction is no more if we replace subsets by subgroups containing the kernel of the homomorphism. In the end, we investigate the role of generalised compressors in relation to direct products.

2. Definition Of Generalised Compressor And Its Relationship With Compressor :

We introduce the concept of 'generalised compressor' in groups and find out a structural relationship between compressor and generalised compressor below.

Def. 6.1 - For any two subsets S_1, S_2 of a group G, the set of all elements $g \in G$ such that

$$(S_1)_g = S_2$$

is called 'S_1-Compressor - S_2' or 'generalised compressor of (S_1, S_2)' in G and is denoted by $C_{S_2}^{S_1}$.

From theorem 1.6 and definition 4.1, it is evident that

(i) If $O(S_1) \neq O(S_2)$, then $C_{S_2}^{S_1} = \phi$

(ii) For $S_1 = S_2 = S$, $C_{S_2}^{S_1} = C_G(S)$.

Theorem 6.1 - Let S_1, S_2 be any two subsets of a group G, then if $g_1 \in C_{S_2}^{S_1}$, $C_{S_2}^{S_1} = g_1 C_G(S_1)$.

Proof. For any element $x_1 \in C_G(S_1)$

$$\begin{aligned} g_1 x_1 S_1 g_1 x_1 &= g_1 x_1 S_1 x_1 g_1 \\ &= g_1 S_1 g_1 \\ &= S_2 \end{aligned}$$

$$\implies g_1 x_1 \in C_{S_2}^{S_1}$$

$$\implies g_1\, C_G(S_1) \subseteq C_{S_2}^{S_1}$$

On the other hand, for any $g \in C_{S_2}^{S_1}$, we have

$$S_2 = g\, S_1 g = g_1\, S_1\, g_1$$

$$\implies g_1^{-1}\, g\, S_1 g\, g_1^{-1} = S_1$$

$$\implies g_1^{-1}\, g\, S_1 g_1^{-1}\, g = S_1$$

$$\implies g_1^{-1}\, g \in C_G(S_1)$$

$$\implies g \in g_1\, C_G(S_1)$$

$$\implies C_{S_2}^{S_1} \subseteq g_1\, C_G(S_1)$$

Hence

$$C_{S_2}^{S_1} = g_1\, C_G(S_1)$$

This proves the theorem.

Cor.6.1 = Let G be a group and O_2 the subgroup of all elements of order 2 in G, then for elements $g_1, g_2 \in G$, $g' \in C_{g_2}^{g_1}$ implies $C_{g_2}^{g_1} = g'\, O_2$.

(Proof is immediate from theorem 6.1, since the compressor of any element of G is O_2) .

Remark : From Cor. 6.1, it follows that for elements g_i, g_i', $i = 1,2$ in a group G,

$$\text{either} \quad C_{g_2}^{g_1} = C_{g_2'}^{g_1'} \quad \text{or} \quad C_{g_2}^{g_1} \cap C_{g_2'}^{g_1'} = \phi$$

Hence there exists a partition of $G \times G$ into mutually disjoint classes K_i such that the pairs (g_1, g_2) and $(g_1', g_2') \in G \times G$ lie in one K_i if, and only if

$$C_{g_2}^{g_1} = C_{g_2'}^{g_1'}$$

Evidently, to every coset $g\, O_2$ of G/O_2 the pairs of elements $(g', g\, g'\, g)$ in $G \times G$, where $g' \in G$ are all elements (g_1, g_2) in $G \times G$ for which

$$C_{g_2}^{g_1} = g\, O_2$$

3. Basic Properties Of Generalised Compressors:

Besides proving some simple properties connected with the product, intersection and union of generalised compressors, we find some properties relating to elements in a generalised compressor and show that an element and its inverse both lie in the generalised compressor of a pair of subsets in a group if, and only if, the square of the element lies in the compressor of the first subset. We also find a condition for equality of generalised compressors of any two pairs of elements in a group.

Theorem 6.2 - Let g_i, $i = 1,2, \ldots n$ be n elements of a group G, and $g_i' \in C^{g_i}_{g_{i+1}}$, $i = 1,2, \ldots n-1$, then we have

(i) $g_1' \cdot g_2' \cdots g_{n-1}' \in C^{g_1}_{g_n}$ (ii) $C^{g_1}_{g_2} = C^{g_1 g'}_{g_2 g'}$ for any $g' \in G$.

Proof : (i) Consider

$$g_1'g_2' \cdots g_{n-1}'(g_1)g_1'g_2' \cdots g_{n-1}' = g_{n-1}' \cdots g_2'(g_1'g_1g_1')g_2' \cdots g_{n-1}'$$

$$= g_{n-1}' \cdots g_2'(g_2)g_2' \cdots g_{n-1}'$$

Continuing in this manner

$$g_1'g_2' \cdots g_{n-1}'(g_1)g_1'g_2' \cdots g_{n-1}' = g_n$$

$$\Longrightarrow g_1'g_2' \cdots g_{n-1}' \in C^{g_1}_{g_n}$$

(ii) For any $g \in C^{g_1}_{g_2}$,

$$g(g_1g')\,g = (gg_1g)g'$$

$$= g_2g'$$

$$\Longrightarrow \quad g \in C^{g_1g'}_{g_2g'}$$

$$\Longrightarrow \quad C^{g_1}_{g_2} \subseteq C^{g_1\,g'}_{g_2\,g'}$$

On the other hand, if $g \in C^{g_1g'}_{g_2g'}$

$$g(g_1g')g = g_2g'$$

$$\Longrightarrow \quad (g\,g_1g)g' = g_2g'$$

$$\Longrightarrow \quad g\,g_1g \quad = g_2$$

$$\Longrightarrow \quad g \in C_{g_2}^{g_1}$$

$$\Longrightarrow \quad C_{g_2g'}^{g_1g'} \subseteq C_{g_2}^{g_1}$$

Consequently

$$C_{g_2}^{g_1} = C_{g_2g'}^{g_1g'}.$$

This completes the theorem.

Note : The above theorem also holds true in case g_i's be subsets in G, however, if in this case g' be also a subset of G, $C_{g_2}^{g_1} \subseteq C_{g_2g'}^{g_1g'}$ holds in general in theorem 6.2 (ii).

Cor. 6.2 - Let g_i, $i = 1,2, \dots n$ be n elements of a group G, then

$$\prod_{i=1}^{n-1} C_{g_{i+1}}^{g_i} \subseteq C_{g_n}^{g_1} = C_{g_2 \cdot g_3 \cdots g_n}^{g_1 \cdot g_2 \cdots g_{n-1}}$$

(Proof follows immediatly from theorem 6.2)

Cor. 6.3 - For any two subsets S_1, S_2 of a group G, $g \in C_{S_2}^{S_1} \cap C_{S_1}^{S_2}$ implies $g^2 \in C_G(S_1) \cap C_G(S_2)$.

(Proof is trivial from theorem 6.2 (i))

Theorem 6.3 - Let g_i, $i = 1,2,3$ be three distinct elements of a group G, then

(i) $g \in C_{g_2}^{g_1}$ if, and only if $g^{-1} \in C_{g_1}^{g_2}$

(ii) $C_{g_2}^{g_1} \cap C_{g_3}^{g_1} = \phi$

(iii) $C_{g_1}^{g_2} \cap C_{g_1}^{g_3} = \phi$

(iv) If $g, g' \in C_{g_2}^{g_1}$ then $g^2 = g'^2$ but not conversely

(v) $g \in C_{g_2}^{g_1}$ if, and only if $g^{-1} \in C_{g_2^{-1}}^{g_1^{-1}}$.

Proof. (i) Clearly, $g \in C_{g_2}^{g_1}$

$$\iff g\, g_1 g = g_2$$

$$\iff g_1 = g^{-1} g_2 g^{-1}$$

$$\iff g^{-1} \in C_{g_1}^{g_2}$$

This proves (i)

(ii) Let $C_{g_2}^{g_1} \cap C_{g_3}^{g_1} \neq \phi$

$\Longrightarrow g\, g_1 g = g_2$ and also $g\, g_1 g = g_3$ for some $g \in G$

$\Longrightarrow \quad g_2 = g_3$

A contradiction to supposition that $g_2 \neq g_3$, hence

$$C_{g_2}^{g_1} \cap C_{g_3}^{g_1} = \phi$$

(iii) Let $$C_{g_1}^{g_2} \cap C_{g_1}^{g_3} \neq \phi$$

$$\Longrightarrow C_{g_2}^{g_1} \cap C_{g_3}^{g_1} \neq \phi \quad \text{by (1)}$$

A contradiction to (ii), hence

$$C_{g_1}^{g_2} \cap C_{g_1}^{g_3} = \phi$$

(iv) Let $g, g' \in C_{g_2}^{g_1}$,

$$\Longrightarrow g\, g_1 g = g' g_1 g' = g_2$$

$$\Longrightarrow g_1 g^2 = g_1 g'^2$$

$$\Longrightarrow g^2 = g'^2$$

Evidently, the converse is false.

This proves (iv)

(v) $$g \in C_{g_2}^{g_1}$$

$$\Longleftrightarrow g\, g_1 g = g_2$$

$$\Longleftrightarrow g^{-1} g_1^{-1} g^{-1} = g_2^{-1}$$

$$\Longleftrightarrow g^{-1} \in C_{g_2^{-1}}^{g_1^{-1}}$$

This completes the theorem.

Note : The (i), (ii), (iii) and (v) of the above theorem hold true in case g_i's be subsets of G.

Theorem 6.4 - For any two subsets S_1, S_2 of a group G, $g \in C_{S_2}^{S_1}$ implies $g^{-1} \in C_{S_2}^{S_1}$ if, and only if $g^2 \in C_G(S_1)$

Proof. Let $g \in C_{S_2}^{S_1}$ implies $g^{-1} \in C_{S_1}^{S1}$. We know, that

$$g^{-1} \in C_{S_2}^{S_1} \Longrightarrow g \in C_{S_1}^{S_2} \quad \text{(Theorem 6.3 (i))}$$

$$\Longrightarrow g \in C_{S_2}^{S_1} \cap C_{S_1}^{S_2}$$

$$\Longrightarrow g^2 \in C_G(S_1) \text{ (cor 6.3)}$$

On the other hand, if $g^2 \in C_G(S_1)$ for $g \in C_{S_2}^{S_1}$, then

$$g \in C_{S_2}^{S_1} = g\, C_G(S_1)$$

$$\Longrightarrow g^{-1} \in g^{-1} C_G(S_1) = g\, C_G(S_1)$$

$$\Longrightarrow g^{-1} \in C_{S_2}^{S_1}$$

This completes the proof.

Theorem 6.5 - Let g_i, g_i', $i = 1,2$ be elements in a group G, then $g \in C_{g_1'}^{g_1} \cap C_{g_2'}^{g_2}$ if, and only if $g_1^{-1} g_1' = g_2^{-1} g_2' = g^2$

Proof. Let

$$g \in C_{g_1'}^{g_1} \cap C_{g_2'}^{g_2}$$

$$\Longrightarrow g\, g_1 g = g_1', \quad g\, g_2 g = g_2'$$

$$\Longrightarrow g_1 g^2 = g_1', \quad g_2 g^2 = g_2'$$

$$\Longrightarrow g_1^{-1}\, g_1' = g_2^{-1}\, g_2' = g^2$$

Conversely, if $g^2 = g_1^{-1} g_1' = g_2^{-1}\, g_2'$

$$\Longrightarrow \quad g\, g_1 g = g_1', \quad g\, g_2 g = g'_2$$

$$\Longrightarrow \quad g \in C_{g_1'}^{g_1} \cap C_{g_2'}^{g_2}$$

This proves the theorem.

Cor.6.4 - Let g_i, g_i', $i = 1,2$ be elements in a group G, and O_2 denotes the subgroup of all elements of order 2, then

$C_{g_1'}^{g_1} = C_{g_2'}^{g_2} = gO_2$ for some $g \in G$ if, and only if

$$g_1^{-1} g_1' = g_2^{-1} g_2' = g^2$$

(Proof of corollary is immediate from theorem 6.5 and Cor.6.1)

4. Nullity Of Generalised Compressors:

We know that if S_1, S_2 be any two subsets of a group such that $O(S_1) \neq O(S_2)$, then $C_{S_2}^{S_1} = \phi$, however, it is evident that the converse is not true in general. We give below a criterion for a group G to have a pair of elements g_1, g_2 such that

$C_{g_2}^{g_1} = \phi$.

Theorem 6.6 - In a group G, there exist elements g_1, $g_2 \in G$ such that $C_{g_2}^{g_1} = \phi$ if, and only if $G \supset G^*$.

Proof. Let for elements $g_1, g_2 \in G$, $C_{g_2}^{g_1} = \phi$. Now, if

$$G^* = G$$

$$\Longrightarrow g_1 G^* = g_1 G = G$$

$$\Longrightarrow g g_1 g = g_2 \quad (\text{ for some } g \in G)$$

$$\Longrightarrow g \in C_{g_2}^{g_1}$$

A contradiction, that $C_{g_2}^{g_1} = \phi$, hence $G^* \subset G$. Conversely, if $G \supset G^*$, then let for no two elements g_1, $g_2 \in G$, $C_{g_2}^{g_1} = \phi$, then for any $g \in G$,

$$C_g^e \neq \phi$$

$$\Longrightarrow g' e g' = g \quad \text{for some } g' \in G$$

$$\Longrightarrow g'^2 = g$$

$$\Longrightarrow G^* = G$$

A contradiction, that $G \supset G^*$, hence there exists atleast a pair (g_1', g_2') of elements in G for which

$$C_{g_2'}^{g_1'} = \phi$$

Thus the proof is complete.

Remark. Evidently, for two subsets S_1, S_2 of the same order in a group, we may have $C_{S_2}^{S_1} = \phi$. We have no general result about the nature of S_1, S_2 in these circumstances. However, we note that

$$C_{S_2}^{S_1} = \phi \not\Rightarrow C_{s_2}^{s_1} = \phi \quad \text{where} \quad s_1 \in S_1,\ s_2 \in S_2$$

For example : Let $G = [a]$, the infinite cyclic group generated by a.

Take $S_1 = \{a, a^3\}$, $S_2 = \{a, a^5\}$

Then, evidently

$$C_{s_2}^{s_1} \neq \phi \quad \text{where } s_1 \in S_1,\ s_2 \in S_2$$

but

$$C_{S_2}^{S_1} = \phi$$

Evidently, $C_{s_2}^{s_1} = \phi$ for every $s_1 \in S_1$, $s_2 \in S$ implies that $C_{S_2}^{S_1} = \phi$. Also it is immediate, that $C_{s_2}^{s_1} = \phi$ for a fixed s_1 (or s_2) in S_1 (or S_2) and arbitrary $s_2 \in S_2 (s_1 \in S_2)$ implies that $C_{S_2}^{S_1} = \phi$.

Theorem 6.7 - Given g_1, g_2 in a group G, $C_{g_2}^{g_1} = \phi$ if, and only if, $g_1 G^* \cap g_2 G^* = \phi$

(Proof is immediate since if $g_1 G^* \cap g_2 G^* \neq \phi$, there exists a $g \in G$ such that $g_1 g^2 = g\, g_1 g = g_2$ and also conversely).

5. Fundamental Mappings Over Generalised Compressors:

We have shown that, under group isomorphism, the image of the generalised compressor of a pair of subsets in a group is equal to the generalised compressor of the pair of image subsets (in the same order) in the image group and have noted that this happens only in a restricted sense in case of group homomorphisms.

Theorem 6.8 - Let ϕ be an isomorphism of a group G onto a group G' and S_1, S_2 be any two subsets of G, then

$$(C_{S_2}^{S_1}) \phi = C_{(S_2)\phi}^{(S_1)\phi}$$

Proof. Let $C_{S_2}^{S_1} = \phi$, then evidently

$$(C_{S_2}^{S_1})\phi \subseteq C_{(S_2)\phi}^{(S_1)\phi}$$

Now, if $C_{(S_2)\phi}^{(S_1)\phi} = \phi$, the theorem holds; but if $C_{(S_2)\phi}^{(S_1)\phi} \neq \phi$

then for $g' \in C_{(S_2)\phi}^{(S_1)\phi}$,

$$g'(S_1\phi)g' = (S_2)\phi$$

$$\implies (g\phi)(S_1\phi)(g\phi) = (S_2)\phi \quad \text{for some } g \in G$$

$$\implies (g S_1 g)\phi = (S_2)\phi$$

$$\implies g S_1 g = S_2 \qquad \text{since } \phi \text{ is 1-1}$$

$$\implies g \in C_{S_2}^{S_1}$$

A contradiction, that $C_{S_2}^{S_1} = \phi$, hence

$$(C_{S_2}^{S_1})\phi = C_{(S_2)\phi}^{(S_1)\phi} = \phi$$

Again, if $C_{S_2}^{S_1} \neq \phi$, then for $g \in C_{S_2}^{S_1}$

$$g\, S_1\, g = S_2$$

$$\begin{aligned}
&\Longrightarrow \quad (g\phi)(S_1\phi)(g\phi) = (S_2)\phi \\
&\Longrightarrow \quad g\phi \in C_{(S_2)\phi}^{(S_1)\phi} \\
&\Longrightarrow \quad C_{(S_2)\phi}^{(S_1)\phi} = (g\phi)\ C_{G'}(S_1\phi) \quad \text{(Theorem 6.1)} \\
&\qquad\qquad\qquad = (g\phi)\ (C_G(S_1)\phi \quad \text{(Theorem 4.16)} \\
&\qquad\qquad\qquad = (g\ C_G(S_1))\phi \\
&\qquad\qquad\qquad = (C_{S_2}^{S_1})\phi \quad \text{(Theorem 6.1)}
\end{aligned}$$

This completes the theorem.

Remark : If ϕ be a homomorphism in the above theorem, then the restricted result

$$(C_{S_2}^{S_1})\phi \subseteq C_{(S_2)\phi}^{(S_1)\phi}$$

holds. However, if S_1, S_2 are subgroups containing the kernel of homomorphism ϕ, then in view of theorem 4.17, the equality again holds.

6. Generalised Compressor In Direct Products:

Theorem 6.9 - Let a group G be direct product of its subgroups G_i, $i = 1,2, \ldots n$ aand S_i, S_i' be subsets of G_i for all i, then

(i) $C_{\prod_{i=1}^{n} S_i'}^{\prod_{i=1}^{n} S_i} = \phi$ if, and only if $C_{S_i'}^{S_i} = \phi$ for some i;

(ii) $\prod_{i=1}^{n} C_{S_i'}^{S_i} = C_{\prod_{i=1}^{n} S_i'}^{\prod_{i=1}^{n} S_i}$

Proof. (i) Let $C_{\prod_{i=1}^{n} S_i'}^{\prod_{i=1}^{n} S_i} = \phi$ and $C_{S_i'}^{S_i} \neq \phi$ for all i, then

if $g_i \in C_{S_i'}^{S_i}$ for all i, we have

$$(g_1 g_2 \ldots g_n)(S_1 S_2 \ldots S_n)(g_1 . g_2 \ldots g_n) = (g_1 S_1 g_1)(g_2 S_2 g_2) \ldots (g_n S_n g_n)$$
$$= S_1' . S_2' \ldots S_n'$$

$$\Longrightarrow g_1 . g_2 \ldots g_n \in C_{\prod_{i=1}^{n} S_i'}^{\prod_{i=1}^{n} S_i}$$

A Contradiction, that $C_{\prod_{i=1}^{n} S_i'}^{\prod_{i=1}^{n} S_i} = \phi$, hence $C_{S_i'}^{S_i} = \phi$ for some i.

Conversely, let $C_{S_i'}^{S_i} = \phi$ for some i and $C_{\prod_{i=1}^{n} S_i'}^{\prod_{i=1}^{n} S_i} \neq \phi$, then if $g \in C^{\prod_{i=1}^{n} S_i}_{\prod_{i=1}^{n} S_i'}$, we have

$$g(S_1 . S_2 \dots S_n)g = S_1' . S_2' \dots S_n'$$

$$\Longrightarrow (g_1 . g_2 \dots g_i \dots g_n)(S_1 . S_2 \dots S_i \dots S_n)(g_1 . g_2 \dots g_i \dots g_n)$$

$$= S_1' . S_2' \dots S_i' \dots S_n', \text{ where}$$

$$g = g_1 . g_2 \dots g_n \text{ and } g_i \in G_i$$

$$\Longrightarrow (g_1 S_1 g_1) \dots (g_i S_i g_i) \dots (g_n S_n g_n) = S_1' . S_2' \dots S_i' \dots S_n'$$

$\Longrightarrow$ $g_i\, S_i\, g_i = S_i'$ for all $i = 1, 2, \dots n$ by uniqueness of representation in $G = \overset{n}{\underset{i=1}{X}} G_i$

$$\Longrightarrow g_i \in C^{S_i}_{S_i'}$$

$$\Longrightarrow C^{S_i}_{S_i'} \neq \phi \text{ for all } i$$

A contradiction, hence $C^{\prod_{i=1}^{n} S_i}_{\prod_{i=1}^{n} S_i'} = \phi$

This proves (i)

(ii) If $C^{S_i}_{S_i'} = \phi$ for some i, then the result follows from (i), hence let

$$C^{S_i}_{S_i'} \neq \phi \quad \text{for any } i$$

Now, if $g_i \in C^{S_i}_{S'_i}$ for all $i = 1,2, \dots n$, then as in (i)

$$g_1 \cdot g_2 \cdots g_n \in C^{\prod_{i=1}^{n} S_i}_{\prod_{i=1}^{n} S'_i}$$

$$\Longrightarrow \prod_{i=1}^{n} C^{S_i}_{S'_i} \subseteq C^{\prod_{i=1}^{n} S_i}_{\prod_{i=1}^{n} S'_i}$$

On the other hand, if for any $g' \in C^{\prod_{i=1}^{n} S_i}_{\prod_{i=1}^{n} S'_i}$, $g' = g'_1 \cdot g'_2 \cdots g'_n$

where $g'_i \in G_i$ for all i then as in (i)

$$g'_i \in C^{S_i}_{S'_i} \qquad \text{for all } i = 1,2, \dots n$$

$$\Longrightarrow g' = g'_1 \cdot g'_2 \cdots g'_n \in \prod_{i=1}^{n} C^{S_i}_{S'_i}$$

$$\Longrightarrow C^{\prod_{i=1}^{n} S_i}_{\prod_{i=1}^{n} S'_i} \subseteq \prod_{i=1}^{n} C^{S_i}_{S'_i}$$

Hence

$$C^{\prod_{i=1}^{n} S_i}_{\prod_{i=1}^{n} S'_i} = \prod_{i=1}^{n} C^{S_i}_{S'_i}$$

This completes the theorem.

It is to be noted here that in the above theorem $C_{S_i'}^{S_i}$ are sets considered in G_i and $C_{\prod_{i=1}^{n} S_i'}^{\prod_{i=1}^{n} S_i}$ in G.

Further, exactly similar results hold in case of external direct product as the following theorem shows:

Theorem 6.10 - Let $G = \underset{i=1}{\overset{n}{X}} G_i$, where G_i's are given arbitrary groups. If S_i, S_i' be any two subsets of G_i for every i, we have

(i) $C_{\underset{i=1}{\overset{n}{X}} S_i'}^{\underset{i=1}{\overset{n}{X}} S_i} = \phi$ if, and only if $C_{S_i'}^{S_i} = \phi$ for some i,

(ii) $C_{\underset{i=1}{\overset{n}{X}} S_i'}^{\underset{i=1}{\overset{n}{X}} S_i} = \underset{i=1}{\overset{n}{X}} C_{S_i'}^{S_i}$.

Proof. Let $C_{\underset{i=1}{\overset{n}{X}} S_i'}^{\underset{i=1}{\overset{n}{X}} S_i} = \phi$ and $C_{S_i'}^{S_i} \neq \phi$ for all i, then if

$g_i \in C_{S_i'}^{S_i}$ for all i,

$$(g_1,\dots g_i,\dots,g_n)(\underset{i=1}{\overset{n}{X}} S_i)(g_1,\dots g_i,\dots,g_n) = \underset{i=1}{\overset{n}{X}} (g_i S_i g_i)$$

$$= \underset{i=1}{\overset{n}{X}} S_i'$$

$$\Longrightarrow (g_1,\ldots,g_i,\ldots,g_n) \in C^{\overset{n}{\underset{i=1}{X}} S_i}_{\overset{n}{\underset{i=1}{X}} S_i'}$$

A contradiction, that $C^{\overset{n}{\underset{i=1}{X}} S_i}_{\overset{n}{\underset{i=1}{X}} S_i'} = \phi$, hence $C^{S_i}_{S_i'} = \phi$ for some i.

Conversely, if $C^{S_i}_{S_i'} = \phi$ for some i, let $C^{\overset{n}{\underset{i=1}{X}} S_i}_{\overset{n}{\underset{i=1}{X}} S_i'} \neq \phi$.

If $g' = (g_1', g_2', g_3', \ldots g_i', \ldots g_n') = (g_i')$ be in $C^{\overset{n}{\underset{i=1}{X}} S_i}_{\overset{n}{\underset{i=1}{X}} S_i'}$, then

$$(g_i')(\overset{n}{\underset{i=1}{X}} S_i)(g_i') = \overset{n}{\underset{i=1}{X}} S_i'$$

$$\Longrightarrow \overset{n}{\underset{i=1}{X}} (g_i' \, S_i \, g_i') = \overset{n}{\underset{i=1}{X}} S_i'$$

$$\Longrightarrow g_i' \, S_i \, g_i' = S_i' \quad \text{for all } i$$

$$\Longrightarrow g_i' \in C^{S_i}_{S_i'}$$

A contradiction, that $C^{S_i}_{S_i'} = \phi$ for some i, hence

$$C^{\overset{n}{\underset{i=1}{X}} S_i}_{\overset{n}{\underset{i=1}{X}} S_i'} = \phi$$

This proves (i)

(ii) If $C_{S_i'}^{S_i} = \phi$ for some i, the result follows from (i); hence let $C_{S_i'}^{S_i} \neq \phi$ for any i. Now, if $g_i \in C_{S_i'}^{S_i}$ for all i then as in (i),

$$(g_1,\ldots,g_i,\ldots g_n) \in C_{\overset{n}{\underset{i=1}{X}} S_i'}^{\overset{n}{\underset{i=1}{X}} S_i}$$

$$\Longrightarrow \overset{n}{\underset{i=1}{X}} C_{S_i'}^{S_i} \subseteq C_{\overset{n}{\underset{i=1}{X}} S_i'}^{\overset{n}{\underset{i=1}{X}} S_i}$$

On the other hand, if $g' = (g_1', g_2', \ldots g_i', \ldots g_n') \in C_{\overset{n}{\underset{i=1}{X}} S_i'}^{\overset{n}{\underset{i=1}{X}} S_i}$, then as in (i)

$$g_i' \in C_{S_i'}^{S_i} \quad \text{for all } i$$

$$\Longrightarrow g' = (g_1', \ldots, g_i', \ldots . g_n') \in \overset{n}{\underset{i=1}{X}} C_{S_i'}^{S_i}$$

$$\Longrightarrow C_{\overset{n}{\underset{i=1}{X}} S_i'}^{\overset{n}{\underset{i=1}{X}} S_i} \subseteq \overset{n}{\underset{i=1}{X}} C_{S_i'}^{S_i}$$

Consequently

$$C_{\overset{n}{\underset{i=1}{X}} S_i'}^{\overset{n}{\underset{i=1}{X}} S_i} = \overset{n}{\underset{i=1}{X}} C_{S_i'}^{S_i}$$

This proves the theorem completly.

Note : The above theorem holds true even in case of complete direct products as can be verified similarily.

7. Concluding Remarks:

In this chapter we have intentionally kept the study of generalised compressors restricted only to subsets of groups and have made no special reference to subgroups. The reason lies in the fact that because of theorem 1.8, which tells us that if a C-transform of a subgroup be again a subgroup it coincides with the subgroup itself; therefore for any two subgroups H_1, H_2 of a group $C_{H_2}^{H_1} = \phi$ unless $H_1 = H_2 = H$(say) in which case it reduces to compressor of H, the study of which has already been made in details in chapter four.

Further, we note that the theorems 6.3(i),(ii),(iii),(v), 6.9 and 6.10 proved here for abelian groups hold true even for non-abelian groups and the same proofs hold in that case also without any change in arguments already supplied.

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CHAPETR - SEVEN

COMPRESSION SERIES AND COMPRESSION CHAINS

1. Introduction: In this chapter, we introduce the concepts of 'Compression series' and 'Compression chains' of a group. We call a finite descending system of subgroups of a group which begins with the group itself and ends in the smallest c.s.c-subgroup of the group, a compression series of the group. We define the isomorphism of two compression series, refinement of a compression series and complete series, then study analogously to that of normal series of a group and obtain analogous of Schreier and Jordon-Holder theorem. We find that the compression series of a group G and normal series of G/G^* are in one-one correspondence, and the corresponding series have the same length. We also establish that any group G having a basis, for which $r_0(G)+r_2(G)<\infty$, has a complete series of length $r_0(G)+r_2(G)$. Further, we define the concept of descending compression chain of a group. In the study of descending compression chains, the most important is the shortest compression chain which is the descending sequence $G \supseteq G^* \supseteq (G^*)^* \supseteq \ldots\ldots\ldots$ of subgroups of G. We show that the shortest compression chain of any subgroup of a group which occurs in some descending compression chain of the group breaks off, if the shortest compression chain of the group breaks off; and both the chains end in the same subgroup. The shortest compression chain of a finitely generated subgroup breaks off if and only if the group is periodic. The important fact to be noted here is that if this chain breaks off in identity subgroup, then all the

elements of the group have orders of powers of 2. The criterion for the shortest compression chain of a group G having basis, for which $r_0(G)+r_2(G)<\infty$, to break off is that every descending compression chain of G should break off. Finally, it is important to observe that groups having all ascending and descending compression chains to be finite have complete series.

2. Concept Of Compression Series:

Def.7.1- A finite system of subgroups of a group G,

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots\ldots\ldots\ldots \supset G_k = G^* \quad \text{—— (i)}$$

beginning with G and ending with the smallest c.s.c-subgroup G^* of G is called a 'Compression series' of G.

It is evident that every G_i in the series (i) is a c.s.c-subgroup of G. Any group G for which $G \supset G^*$ has a compression series. If H be any c.s.c-subgroup of G, distinct from G and G^*, then

$$G \supset H \supset G^*$$

is a compression series. This means that for any given c.s.c-subgroup of G other than G and G^*, there exists compression series that passes through it.
The factor groups

$$G/G_1,\ G_1/G_2,\ \ldots\ldots,\ G_{k-1}/G^*$$

are called the factors of the compression series (i). The number of factors is called the length of the series, the length of series (i) is, for example, k.

3. Properties Of Compression Series:

Def. 7.2 - Any two compression series of a group are isomorphic if there exists a 1-1 correspondence between the factors of the series such that the paired factors be isomorphic.

Def. 7.3 - For any group G a compression series

$$G \supset H_1 \supset H_2 \quad \ldots\ldots\ldots \supset H_l = G^*$$

is called a 'refinement' of the compression series

$$G \supset G_1 \supset G_2 \supset \ldots\ldots\ldots \supset G_k = G^*$$

if every subgroup G_i coincides with one of the subgroups H_j.

Evidently, every compression series is a refinement of itself and the length of a compression series is less than or equal to the length of its refinement. We have, here, an analogue of Schreier theorem on isomorphic refinements.

Theorem 7.1 - Any two compression series of a group G have isomorphic refinements.

Proof. Let

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots\ldots\ldots \supset G_l = G^* \qquad (i)$$

$$G = H_0 \supset H_1 \supset H_2 \supset \ldots\ldots\ldots \supset H_m = G^* \qquad (ii)$$

be any two compression series of G. Consider

$$G_{ij} = (G_{i-1} \cap H_j)G_i, \qquad j = 0,1,2, \dots m.$$

$$H_{ij} = (H_{j-1} \cap G_i)H_j, \qquad i = 0,1,2, \dots \ell.$$

Evidently, G_{ij} and H_{ij} are c.s.c-subgroups of G, and for all $i = 1,2, \dots \ell$, $j = 1,2, \dots m$, we have

$$G_{i-1} = G_{i0} \supseteq G_{i,j-1} \supseteq G_{ij} \supseteq G_{im} = G_i$$

$$H_{j-1} = H_{0j} \supseteq H_{i-1,j} \supseteq H_{ij} \supseteq H_{\ell j} = H_j$$

Thus, we may now obtain refinements of (i) and (ii) by inserting G_{ij}, $j = 1,2 \dots m$ between G_{i-1} and G_i for all $i = 1,2, \dots \ell$ and H_{ij}, $i = 1,2, \dots \ell$ between H_{j-1} and H_j for all $j = 1,2, \dots m$ respectively. These are in general compression series with repititions. By Zassenhaus lemma, $G_{i,j-1}/G_{ij} \cong H_{i-1,j}/H_{ij}$, hence these refinements are isomorphic. Finally, it is also clear that whenever $G_{i,j-1} = G_{ij}$ then also $H_{i,j-1} = H_{ij}$ and therefore we can eliminate simultaneously the repititions in these refinements of series (i) and (ii) without effecting isomorphism.

This completes the proof.

Theorem 7.2 - For any group G, there exists a 1-1 correspondence between compression series of G and normal series of G/G*, moreover the corresponding series have the same length.

Proof. Let

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots\cdots \supset G_k = G^*$$

be a compression series of G. If we put $G/G^* = \overline{G}$ and $G_i/G^* = \overline{G_i}$ for all $i = 1, 2, \ldots k$ then

$$G/G^* = \overline{G} \supset \overline{G}_1 \supset \overline{G}_2 \supset \cdots\cdots \supset \overline{G}_k = E \in G/G^*$$

is a normal series of G/G^* since G is abelian. Conversely, if

$$G/G^* = \overline{H}_0 \supset \overline{H}_1 \supset \cdots\cdots \supseteq \overline{H}_\ell = E$$

be any normal series of G/G^*, let H_i denotes the inverse image of $\overline{H}_i$ under the natural homomorphism of G onto G/G^* for all $i = 0, 1, \ldots, \ell$, then

$$G = H_0 \supset H_1 \supset H_2 \supset \cdots\cdots \supset H_\ell = G^*$$

is a compression series of G. Thus we note that to each compression series of G there corresponds a normal series of G/G^* of the same length and vice versa, moreover, this correspondence is 1-1 since there exists a 1-1 correspondence between the subgroups of G containing G^* and subgroups of G/G^* under the natural homomorphism of G onto G/G^*.

This completes the theorem.

Note: The above theorem suggests another proof of theorem 7.1, in view of Schreiers theorem.

Def. 7.4 - A compression series of a group G that has no refinement other than itself is called a complete series.

Let

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots\ldots \supset G_k = G^*$$

be a complete series of a group G then every G_i, $i = 1,2, \ldots k$ is a maximal c.s.c-subgroup of G in G_{i-1}. All factors G_{i-1}/G_i of the series have identity subgroup to be the maximal c.s.c-subgroup other than itself i.e. all elements of G_{i-1}/G_i are of order 2 and moreover $[G_{i-1}: G_i] = 2$. Conversely, every compression series of G all of whose factors have identity subgroup to be the maximal c.s.c-subgroup of the group other than itself cannot be further refined i.e. such a series is a complete series. Evidently, every compression series isomorphic to a complete series is a complete series. This fact gives a theorem equivalent to Jordan Holder theorem for compression series.

Theorem 7.3 - If a group G has a complete series then anytwo complete series of G are isomorphic.

(Proof is immediate. from theorem 7.1)

Lemma 7.1 - Let G be a finitely generated group such that $r_0(G) + r_2(G) = m$, then G has a complete series of length m.

Proof. Let $\{g_1, g_2, \ldots g_m\}$ denotes the set of all elements of infinite order and of order $2^k (k\geq 1)$ in a basis of G containing elements of prime power and / or infinite order. We define

$$G_i = [G^*, g_1, g_2 \ldots, g_i] \text{, The subgroup generated by } G^* \text{ and } g_1, g_2, \ldots g_i.$$

for all $1 \leqslant i \leqslant m$, then

$$[G_i : G^*] = 2^i \qquad \text{(Theorem 3.6)}$$

Now, it is evident that

$$G = G_m \supset G_{m-1} \supset \text{-------} \supset G_2 \supset G_1 \supset G_0 = G^*$$

is a compression series of G of length m. This is a complete. series of G, since

$$[G_i : G_{i-1}] = 2 \quad \text{for all } i = 1, 2, \ldots, m.$$

Hence the proof is complete.

Note: In view of the remark following the Theorem 3.6, the result proved above holds true for anygroup G having a basis for which $r_0(G) + r_2(G) < \infty$.

Theorem 7.4 - Let G be a group having a basis and $r_o(G) + r_2(G) < \infty$ then any complete series of G has length $r_o(G) + r_2(G)$.

(Proof of theorem is immediate from the note following lemma 7.1 and theorem 7.3).

Theorem 7.5 - A group G has a complete series if and only if G/G^* has a composition series.

(Proof is evident from theorem 7.2).

Theorem 7.6 - Let

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots\ldots \supset G_k = G^*$$

be a complete series of a group G. Then every subgroup H of G such that $G^* \cap H = H^*$, has a compression series whose factors are isomorphic to subgroups of distinct factors of the complete series.

Proof. We define

$$H_i = H \cap G_i \quad \text{for all } i = 0,1,2, \dots k.$$

By corollary 3.1, every H_i is a c.s.c-subgroup of H. Further, it is clear that $H_{i-1} \supseteq H_i$ for all i = 1,2, ... k. Now, since $H_{i-1} \supseteq H_i$, $G_{i-1} \supseteq G_i$ we have by Zassenhaus lemma

$$H_{i-1}/H_i \cong G_i H_{i-1}/H_i$$

Here $G_{i-1} \supseteq G_i H_{i-1} \supseteq G_i$, hence the factor group H_{i-1}/H_i is isomorphic to a subgroup of the factor group H_{i-1}/H_i . Thus the series

$$H = H_0 \supseteq H_1 \supseteq H_2 \supseteq \dots\dots \supseteq H_k = H^*$$

after deletion of repititions is the required compression series of H.

This completes the theorem.

4- Compression Chains :

Def. 7.5 - A descending sequence of subgroups of a group G

$$G = G_0 \supset G_1 \supset G_2 \supset \dots\dots \supset G_n \supset \dots\dots \qquad \text{(i)}$$

is called a 'descending compression chain' of G if every subgroup G_n, n = 1,2 ..... is a proper c.s.c-subgroup of G_{n-1}.

The descending compression chain (i) may be finite or countably infinite. In the former case, we say that the chain breaks off. If we define $(G^*)^* = G^{2*}$, $(G^{2*})^* = G^{3*}$ and in general $(G^{(n-1)*})^* = G^{n*}$, then we obtain a descending sequence of subgroups of G

$$G = G^{0*} \supseteq G^* \supseteq G^{2*} \supseteq \ldots \supseteq G^{n*} \supseteq \ldots\ldots$$

which is clearly a descending compression chain of G. This we call 'shortest compression chain' of G. This chain may be continued transfinitely, if we define for any limit ordinal number α, $G^{\alpha*}$ to be the intersection of $G^{\beta*}$, $\beta < \alpha$ and $G^{\alpha*} = (G^{(\alpha-1)*})^*$ if α is not a limit ordinal number. Evidently, for some ordinal number γ whose cardinal number is not greater than that of the power of the group itself, we have

$$G^{\gamma*} = G^{(\gamma+1)*}$$

These subgroups $G^{\alpha*}$ obtained thus are fully invariant in G, since obviously the smallest c.s.c-subgroup of a group is fully invariant and the property of being fully invariant is transitive and is also preserved under operation of intersection.

Theorem 7.7- If the shortest compression chain of a group G breaks off, then the shortest compression chain of any of its subgroup H that occurs in some compression chain of G breaks off; and both the chains end with the same subgroup. The length of the shortest compression chain of G is greater than or equal to the length of the corresponding chain for H.

Proof. Let the shortest compression chain of G breaks off at the n^{th} stage, then

$$G^{n*} = G^{j*} \quad \text{for all } j \geqslant n$$

Now, let H be any subgroup of G that occurs in a compression chain of G and let that chain be

$$G = H_0 \supset H_1 \supset H_2 \supset \ldots\ldots\ldots \supset H_m \supset \ldots\ldots\ldots$$

Then, evidently

$$H_i \supseteq G^{i*} \quad \text{for all } i \geqslant 1$$

$$\Longrightarrow \quad H_i \supseteq G^{n*} \quad \text{for all } i \geqslant 0$$

$$\Longrightarrow \quad H \supseteq G^{n*} \quad \text{since H is some } H_i$$

$$\Longrightarrow \quad H^{n*} \supseteq G^{n*}$$

But, it is clear that

$$G^{n*} \supseteq H^{n*}$$

$$\Longrightarrow \quad H^{n*} = G^{n*}$$

$$\Longrightarrow \quad H^{n*} = H^{i*} \quad \text{for all } i \geqslant n.$$

$\Longrightarrow$ The shortest compression chain of H breaks off with the last elements equal to G^{n*}

This proves the theorem completly.

Theorem 7.8 - Let the shortest compression chain of a group G breaks off at the m^{th} stage. Then a descending compression chain

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots\cdots \supset G_k \supset \cdots\cdots \supset G_n \supset \cdots\cdots\cdots \quad (i)$$

of G breaks off if for a subgroup G_k in (i), (1) $G_k^* = G^{m*}$ and G_k has a complete series, or (2) $G_k = G^{m*}$.

Proof. (1) Evidently, in the descending compression chain (i), we have

$$G \supset G_1 \supseteq G^*$$
$$G_1 \supset G_2 \supseteq G_1^* \supseteq G^{2*}$$
$$G_2 \supset G_3 \supseteq G_2^* \supseteq G_1^{2*} \supseteq G^{3*}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$G_i \supset G_{i+1} \supseteq G_i^* \supseteq G_{i-1}^{2*} \supseteq \cdots \supseteq G^{(i+1)*}$$

Now, if k be the smallest integer for which G_k has a complete series , and that

$$G_k^* = G^{m*}$$

$\Longrightarrow$ $G_i \supseteq G^{m*} = G_k^*$ for all $i \geqslant k$, since $G^{m*} = G^{i*}$ forall $i \geqslant m$.

$\Longrightarrow$ $G_k \supset G_{k+1} \supset \cdots\cdots$ is a part of a compression series of G_k.

Hence, since G_k has a complete series, the descending chain (i) breaks off. This proves the first part.

For the 2nd part, let us have

$$G_k = G^{m*} \quad \text{for some } G_k \text{ in (i)}$$

$\Longrightarrow$ $G_i = G^{m*}$ for all $i \geqslant k$, since $G^{m*} = G^{i*}$ for all $i \geqslant m$.

$\Longrightarrow$ $G_k = G_i$ for all $i \geqslant k$.

Hence the descending chain (i) breaks off.

This proves the theorem completly.

Theorem 7.9 - Let G be a finitely generated group, then the shortest compression chain of G breaks off if, and only if, G be periodic.

Proof. Let $B = \{g_1, g_2 \ldots, g_n\}$ be a basis of G. If G be periodic, then every g_i $(1 \leq i \leq n)$ is of finite order. From corollary 3.4,

$$G^* = [g_1^2] \times [g_2^2] \times \ldots\ldots\ldots \times [g_n^2]$$

Furtherm from theorem 3.4

$$G^{2*} = [g_1^4] \times [g_2^4] \times \ldots\ldots \times [g_n^4]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$G^{m*} = [g_1^{2^m}] \times [g_2^{2^m}] \times \ldots\ldots \times [g_n^{2^m}]$$

.... and so on.

It is clear that for some number $k \geq 0$, $O(g_i^{2^k}) =$ odd, for all g_i's and consequently

$$G^{k*} = G^{i*} \quad \text{for all } i \geq k$$

Hence the shortest compression chain of G breaks off. Conversely, let the shortest compresshin chain of G breaks off. Now, if G is not periodic, then for some $1 \leq m_1 \leq n$, we have $O(g_{m_1}) = \infty$. Here, it is immediate that

$$G^{m'*} \neq G^{n'*} \quad \text{for any } (m' \neq n')$$

because, $[g_{m_1}^{2^{m'}}] \neq [g_{m_1}^{2^{n'}}]$ for any two integers $m', n' \geq 0 (m' \neq n')$.

A contradiction to supposition that the shortest compression chain of G breaks off. This completes the proof.

Note. The above theorem also holds true for any group G having a basis orders of whose elements admit an upper bound. It also follows clearly from the theorem that the shortest compression chain of a finitely generated group breaks off at identity if, and only if, every element of the group be of order $2^n (n \geq 0)$.

Theorem.7.10 - Let G be a group having basis such that $r_0(G) + r_2(G) < \infty$. The shortest compression chain of G breaks off if and only if every descending compression chain of G breaks off.

Proof. If every descending compression chain of G breaks off, then clearly the smallest compression chain of G breaks off, since it is a descending compression chain of G. To prove the converse, let

$$G = H_0 \supset H_1 \supset H_2 \supset \ldots\ldots \supset H_n \supset \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

be a descending compression chain of G. Evidently

$$H_1 \supseteq G^*, \quad H_2 \supseteq G^{2*}, \ \ldots\ldots, \ H_n \supseteq G^{n*}, \ldots\ldots$$

Now, if the shortest compression chain of G breaks off at m^{th} stage, we have $G^{m*} = G^{i*}$ for all $i \geq m$, hence $H_i \supseteq G^{m*}$ for all $i \geq 0$. We know that $r_0(G) + r_2(G) < \infty$, hence by the remark following theorem 3.6

$$[G : G^*] = 2^{r_0(G) + r_2(G)} < \infty$$

Further, it is clear that any subgroup H of G has a basis and $r_0(H) + r_2(H) \leq r_0(G) + r_2(G) < \infty$ hence

$$[G^{i*} : G^{(i+1)*}] = 2^{r_0(G^{i*}) + r_2(G^{i*})} < \infty, \text{ for all } i = 1, 2, \ldots, m-1$$

Consequently

$$[G : G^{m*}] < \infty$$

$\Longrightarrow$ The chain (i) breaks off since $H_i \geq G^{m*}$ for all $i \geqslant 0$.

This completes the theorem.

It is to be noted that even if a group G has a complete series, a subgroup of G may or may not have it. However, the following theorem holds:

Theorem 7.11 - Let G be a group having basis such that G has a complete series, then every subgroup H of G has a complete series.

Proof. Since G has a basis, every subgroup H of G has a basis. Further, since G has a complete series, it follows, in view of the remark following theorem 3.6, that

$$[G : G^*] = 2^{r_0(G) + r_2(G)} < \infty$$

$$\Longrightarrow r_0(G) + r_2(G) < \infty$$

$$\Longrightarrow r_0(H) + r_2(H) \leqslant r_0(G) + r_2(G) < \infty$$

Hence, from the remark following lemma 7.1, it follows that H has a complete series.

This proves the theorem.

Def. 7.6 - An ascending sequence of subgroups of a group G

$$G_1 \subset G_2 \subset G_3 \subset \ldots\ldots\ldots\ldots \subset G_n \subset \ldots\ldots\ldots\ldots$$

is called an 'ascending compression chain' of G if every G_i, $i = 1,2, \ldots$ is a proper c.s.c-subgroup of G_{i+1} and all the subgroups G_i's occur in some descending compression chain of G.

We note here, that in general if the shortest compression chain of a group G breaks off, then every ascending or descending compression chain of G need not break off. For example

Let $G = \underset{i=1}{\overset{\infty}{X}} [a_i]$ where $a_i^2 = e$ for all i

$$G_n = \underset{i=1}{\overset{\infty}{X}} [a_{i+n}] , \quad H_n = \underset{i=1}{\overset{n}{X}} [a_i] \quad \forall \ n = 1,2, \ldots$$

Then, evidently

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots\ldots \supset G_n \supset \ldots\ldots \quad \text{(i)}$$

and

$$H_1 \subset H_2 \subset H_3 \subset \ldots\ldots \subset H_n \subset \ldots\ldots \quad \text{(ii)}$$

are the descending and ascending compression chains of G respectively. These are infinite; but clearly the shortest compression chain of G breaks off at the Ist stage itself. However, the following result holds :

Theorem 7.12 - A group G has a complete series if all its ascending and descending compression chains break off.

Proof. The condition that every ascending compression chain of G breaks off implies that every c.s.c-subgroup H of G contains a c.s.c-subgroup H' of G such that $H' \subset H$ is maximal subgroup of this type. In particular, $G = H_0$ has a maximal proper c.s.c-subgroup. Let, subgroups.

$$G = H_0, H_1, H_2, \dots, H_n$$

have been selected such that every H_i is a proper c.s.c-subgroup of G and $H_{i+1} \subset H_i$ is a maximal proper subgroup of this type. In case $H_n \neq G^*$, we can choose a c.s.c-subgroup H_{n+1} of G such that $H_{n+1} \subset H_n$ be maximal. Since every descending compression chain of G breaks off, we must arrive after a finite number of steps at G^*, thus we obtain a complete series of G.

This proves the theorem.

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Part II
(Non-Abelian Groups)

CHAPTER - EIGHT

COMPRESSED - TRANSFORMS

1. Introduction : This study of compressed transforms is in continuation to its study in chapter- one with the difference that the definitions 1.1 and 1.2 are studied now in non-abelian groups. In chapter one, we had proved that the 'compression' is an equivalence relation in an abelian group (Theorem 1.1); but we find that it is not so in non-abelian groups. 'Compression' is only reflexive and symmetric but not necessarily transitive in non-abelian cases. We, therefore, deduce a criterion for the 'compression' to be an equivalence relation over elements of a group, and point out that this equivalence is not equivalent to its equivalence in general. We do not say anything about the non-abelian groups in which the 'Compression' is an equivalence relation.

In the sequel, by groups, we shall mean non-abelian groups unless otherwise mentioned. Besides finding some properties of C-transforms of subgroups in relation to cosets and conjugates of subgroups, and giving some basic results about C-transforms of subsets, we study the properties of collections of C.transforms. In this connection, we establish that in a group G. such that $O\{g^2 \mid g \in G\} < \infty$, all elements with finite compressions in G form a characteristic subgroup and that, in such a group, the condition for an element to have finite number of conjugates in equivalent to have finite number of compressions (Theorem 8.12).
We also deduce a criterion for an element of a group, which

has finite compressions with respect to elements of a subgroup of finite index, to have finite compressions with respect to elements of the group (Theorem 8.13).

2. The Problem Of 'Compression' As An Equivalence Relation:

In the study of the relation of ' compression' in non-abelian groups, we find that it is not in general transitive even if it be transitive over the elements of the group. We have, however, investigated a criterion under which compression over elements of a group is an equivalence relation. This result is only a restricted one. We have not been able to see whether a group in which compression is an equivalence relation is abelian ? The problem in this respect is open.

Theorem 9.1 - The relation of 'compression' is reflexive and symmetric in a group G but not necessarily transitive.

Proof. The proof that the relation of 'compression' is reflexive and symmetric is immediate from theorem 1.1. To show that compression is not necessarily transitive, we consider the following example:

Let $G = [a,b]$ with $a^2 = e$, $b^5 = e$, $bab = a$

Evidently $G = \{e, a, b, b^2, b^3, b^4, ab, ab^2, ab^3, ab^4\}$

We easily have

$$\text{(i)} \quad x\,b^4x \begin{cases} = e & \text{for } x = b^3 \in G \\ \neq e & \text{for } x(\neq b^3) \in G \end{cases}$$

$$(ii) \quad \left.\begin{array}{l} x\,ab^2x \neq e \\ x\,ab\,x \neq e \end{array}\right\} \quad \text{for any } x \in G$$

We take $S_1 = \{a, e, ab\}$, $S_2 = \{a, b, ab\}$ and $S_3 = \{ab, b^4, ab^2\}$

Then

$$(S_3)_{ab} = S_2, \ (S_2)_{b^2} = S_1 \text{ but } (S_3)_x \neq S_1 \text{ for any } x \in G$$

which can be readily checked by (i) and (ii). This proves that

$$S_3 \xrightarrow{C} S_2, \quad S_2 \xrightarrow{C} S_1 \ \not\Rightarrow \ S_3 \xrightarrow{C} S_1$$

We know that the compression is an equivalence relation in an abelian group. Naturally a question arises 'What can we say for a group in which compression is an equivalence relation?. We know nothing in general, except in the case it be equivalent over its elements. Apparently, one may be led to feel that in such a case, the group may be abelian but even this is not known. We, however, enter into following investigations:

Theorem 8.2 - If the relation of compression is equivalent over elements of a group, then the group is not necessarily abelian.

Proof. We prove the statement by contradiction. Consider the following example:

Let $G = S_3$, the symmetric group of degree 3 whose elements are denoted as below:

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \alpha_1 = (2\ 3), \alpha_2 = (1\ 2), \alpha_3 = (1\ 3),$$

$$\alpha_4 = (1\ 2\ 3), \alpha_5 = (1\ 3\ 2)$$

Then, we have

$$\{x \alpha_1 x \mid x \in G\} = \{\alpha_1, \alpha_2, \alpha_3\}$$

$$\{x \alpha_2 x \mid x \in G\} = \{\alpha_1, \alpha_2, \alpha_3\}$$

$$\{x \alpha_3 x \mid x \in G\} = \{\alpha_1, \alpha_2, \alpha_3\}$$

$$\{x \alpha_4 x \mid x \in G\} = \{e, \alpha_4, \alpha_5\}$$

$$\{x \alpha_5 x \mid x \in G\} = \{e, \alpha_4, \alpha_5\}$$

$$\{x\, e\, x \mid x \in G\} = \{e, \alpha_4, \alpha_5\}$$

From the above tables it is immediate that the relation of 'compression' is equivalent over elements of G, however, G is non-abelian.

This proves the assertion.

Theorem 8.3 - If the relation of compression is transitive over subsets of a group G, it is transitive over elements of G but not conversely.

Proof. Necessity is obvious. To prove that the converse is false, consider $G = S_3$, the symmetric group of degree 3. Evidently, as in theorem 8.2, the compression is transitive over elements of G but if we take the following subsets with the notations of theorem 8.2

$$S_1 = \{\alpha_2, \alpha_3, \alpha_4\}, \quad S_2 = \{\alpha_2, \alpha_3, \alpha_5\}, \quad S_3 = \{\alpha_1, \alpha_2, \alpha_4\}$$

we note that

$$(S_1)_{\alpha_5} = S_2, \quad (S_2)_{\alpha_2} = S_3 \text{ but } (S_1)_x \neq S_3 \text{ for any } x \in G.$$

Thus

$$S_1 \xrightarrow{C} S_2, \quad S_2 \xrightarrow{C} S_3 \not\Longrightarrow S_1 \xrightarrow{C} S_3.$$

The following theorem gives a necessary and sufficient condition for compression to be an equivalence relation over elements of a group.

Theorem 8.4 - Let in a group G, $\bar{g} = \{x\, gx \mid x \in G\}$ for every $g \in G$, then the relation of 'compression' is transitive over elements of G if, and only if, $\bar{g}_1 = \bar{g}_2$ or $\bar{g}_1 \cap \bar{g}_2 = \phi$ for any two $g_1, g_2 \in G$.

Proof. To prove that the condition is sufficient, let $g_1, g_2, g_3 \in G$ such that

$$x g_1 x = g_2, \quad y g_2 y = g_3 \quad \text{where } x, y \in G$$

$$\Longrightarrow \quad g_2 \in \bar{g}_1, \quad g_3 \in \bar{g}_2$$

$$\Longrightarrow \quad \bar{g}_1 \cap \bar{g}_2 \neq \phi, \quad \bar{g}_2 \cap \bar{g}_3 \neq \phi$$

$$\Longrightarrow \quad \bar{g}_1 = \bar{g}_2 = \bar{g}_3$$

$$\Longrightarrow \quad g_3 \in \bar{g}_1$$

$$\Longrightarrow \gamma g_1 \gamma = g_3 \quad \text{for some } \gamma \in G$$

Thus the 'compression' is transitive over elements of G. Conversely, if the 'compression' be transitive over elements of G, then let for $g_1, g_2 \in G$

$$\bar{g}_1 \cap \bar{g}_2 \neq \phi$$

$$\Longrightarrow x g_1 x = y g_2 y \quad \text{for some } x, y \in G$$

Also since $y^{-1}(y g_2 y)y^{-1} = g_2$

$$\Longrightarrow \gamma g_1 \gamma = g_2 \quad \text{where } \gamma \in G \text{ (Transitivity)}$$

Now, for any $x \in G$

$$x g_2 x = g$$

$$\Longrightarrow \gamma_1 g_1 \gamma_1 = g \quad \text{for some } \gamma_1 \in G \text{ (Transitivity)}$$

$$\Longrightarrow \bar{g}_2 \subseteq \bar{g}_1$$

Similarily, since

$$\gamma g_1 \gamma = g_2$$

$$\Longrightarrow g_1 = \gamma^{-1} g_2 \gamma^{-1}$$

We have again as above, that

$$\bar{g}_1 \subseteq \bar{g}_2$$

Hence

$$\bar{g}_1 = \bar{g}_2$$

Thus the theorem is proved.

3. Properties Of C-Transforms Of Subgroups And Subsets:

We, here, find mainly some properties of C-transforms of subgroups connected with conjugates and cosets of subgroups. We get a criterion for equality of two C-transforms of a normal subgroup. We also find that an element belongs to the centralizer of a subset if and only if it belongs to the centralizer of the C-transform of the subset with respect to the element. Finally, we note that in a direct decomposition of a group, the product of C-transforms of subsets of its direct factors with respect to elements of the corresponding direct factors is the C-transform of the product set with respect to the product element of the group.

Theorem 8.5 - Let H_1, H_2 be two subgroups of a group G and for some $x \in G$, $x^{-1}H_1x = H_1$, $(H_1)_x = H_2$ then $H_1 = H_2$

Proof. Given $(H_1)_x = H_2$

$$\Longrightarrow \quad xex = x^2 \in H_2$$

Further

$$(H_1)_x = H_2$$

$$\Longrightarrow \quad H_1 = (H_2)_{x^{-1}}$$

$$\Longrightarrow \quad x^{-1}H_1x = (H_2)x^{-1}$$

$$\Longrightarrow \quad H_1 = H_2.x^{-2}$$

$$= H_2 \quad \text{since} \quad x^2 \in H_2$$

This completes the theorem.

Cor. 8.1 - If a C-transform of a normal subgroup be a subgroup then it must coincide with the subgroup itself.

Note: Evidently, for any subgroup H of a group G and $x \in G$, $H_x = x^{-1} Hx$ if and only if $x^2 \in H$.

Theorem 8.6 - For any normal subgroup H of a group G and $x, y \in G$, $H_x = H_y$ if, and only if, for every $h \in H$, there exists $h' \in H$ such that $y^{-1} x h x y^{-1} = h'$.

Proof. Necessity is obvious. To prove the sufficient part, let $h \in H$ be arbitrary, then

$$y^{-1} x h x y^{-1} = h' \qquad \text{where } h' \in H$$
$$\Longrightarrow \quad H_x \subseteq H_y$$
$$\Longrightarrow \quad H_x = H_y$$

Hence the theorem is complete.

Note. In general, the above condition is only necessary for an arbitrary subset S of G; however, the theorem holds true in view of theorem 1.6 if S be finite.

Theorem 8.7 - Let H be a subgroup of a group G and $x \in G$, then

$$H_x \cap x^{-1} H x = \phi \quad \text{if, and only if, } x^2 \notin H.$$

Proof. Let $H_x \cap x^{-1} H x = \phi$. Suppose, in contradiction, $x^2 \in H$

$$\Longrightarrow \quad x^2 = h' \qquad \text{where } h' \in H$$

$$\Longrightarrow \quad x = x^{-1} h'$$

$$\Longrightarrow x h x = x^{-1} h' h x \quad \text{for every } h \in H$$

$$\Longrightarrow H_x \cap x^{-1} H x \neq \phi$$

A contradiction to the assumption.

Conversely, if $x^2 \notin H$

$$H_x \cap x^{-1} H x \neq \phi$$

$$\Longrightarrow x h x = x^{-1} h' x \qquad h, h' \in H$$

$$\Longrightarrow x^2 = h' h^{-1} \in H$$

Hence the theorem follows.

Cor. 8.2 - For any normal subgroup H of a group G and any $x \in G$, $H_x \cap H \neq \phi$ implies $H_x = H$.

Theorem 8.8 - Let H be a subgroup of a group G and $x, y \in G$, then

(i) $x H = y H$ if and only if $x y \in H_y$ or $y x \in H_x$

(ii) $H x = H y$ if and only if $x y \in H_x$ or $y x \in H_y$

Proof. (i) We have

$$x H = y H$$

$$\Longleftrightarrow x \in y H$$

$$\Longleftrightarrow x y \in H_y$$

The rest of the proof follows by symmetry. Similarily we can prove (ii)

This completes the theorem.

Theorem 8.9 - Let S be a subset of a group G, then $x \in Z(S)$ if and only if $x \in Z(S_x)$.

Proof. If $x \in Z(S)$, then

$$x s = s x \qquad \text{for every } s \in S$$

$$\Longrightarrow x(x sx) = (x sx)x$$

$$\Longrightarrow x \in Z(S_x)$$

The converse follows easily by reversing the argument.

This proves the result.

Theorem 8.10 - Let a group G^* be direct product of its subgroups G_i ; $i = 1,2, \dots n$ and S_i denotes a subset of G_i for every i, then if $x_i \in G_i$

$$\prod_{i=1}^{n} (S_i)_{x_i} = \left(\prod_{i=1}^{n} S_i\right)_{\prod_{i=1}^{n} x_i}$$

Proof : (The proof is simple in view of the definition of direct product.)

4. Collection Of C-Transforms:

We find that the set of all elements which have finite number of compressions in a group G, squares of whose elements from a finite set, is a characteristic subgroup and that in such a group G, an element lies in a finite class of conjugate elements if and only if its compressions in the group form a finite set. We also give a criterion for an element of a group having finite number of compressions with respect to elements of a subgroup of finite index to have finite number of compressions in the group.

Theorem 8.11 - Let for a group G, $/G_1 = \{x^2 | x \in G\}$ be finite, then the set H of all elements in G whose compressions with respect to elements of G form finite sets, is a characteristic subgroup of G.

Proof. Given $O(G_1) < \infty$

$$\implies e \in H$$

Now, let $h_1, h_2 \in H$, then for any $x \in G$,

$$x^{-1} h_1 x = x^{-2} \cdot x h_1 x$$

$$\implies O\{x^{-1}h_1 x \mid x \in G\} < \infty$$

Hence

$$x(h_1h_2)x = (x h_1 x^{-1})(x h_2 x)$$

$$\implies O\{x(h_1h_2)x \mid x \in G\} < \infty$$

$$\implies h_1 h_2 \in H$$

Also

$$x h_2^{-1} x = (x^{-1} h_2 x^{-1})^{-1}$$

$$\Longrightarrow h_2^{-1} \in H$$

Thus H is a subgroup of G.

Finally, if η be any antomorphism of G and $x, y \in G$, then

$$x h_1 x = y h_1 y \iff (x\eta)(h_1\eta)(x\eta) = (y\eta)(h_1\eta)(y\eta)$$

$$\Longrightarrow H = H\eta$$

This completes the theorem.

Remark : If we drop the condition of finiteness of G_1 in the above theorem, then clearly H is not a subgroup and also conversely; but the question whether in such a case $H \neq \phi$ remains open. Thus, if H be a subgroup, then G is periodic, but the converse is evidently false in general.

Theorem 8.12 - Let in a group G, $G_1 = \{x^2 \mid x \in G\}$ is finite, then the set of all compressions of an element of G with respect to elements in G is finite if and only if the conjugate class of the element be finite.

Proof. The proof follows immediatly in view of the identity :

$$x^2 \cdot x^{-1} g x = x g x \qquad \text{where } x, g \in G.$$

Cor. 8.3 - Let in a group G, $G_1 = \{x^2 \mid x \in G\}$ be finite, then every conjugate class of elements in G is finite if and only if every element of G has finite compressions in G.

Theorem 8.13 - Let H be a subgroup of finite index n in a group G, then any element of G whose compressions with respect to elements in H form a finite set, has also a finite number of compressions with respect to elements in G if and only if for any set $\{x_i\}_{i=1}^{n}$ of coset representatives of H in G $O\{h^{-1} x_i h \mid h \in H\} < \infty$ for all i.

Proof. Let for some $g \in G$,

$$O\{h\, gh \mid h \in H\} < \infty$$

Then, for any $x \in G$

$$x\, gx = x_i\, h\, (g) x_i\, h \quad \text{for some } i, \text{ such that } 1 \leq i \leq n \text{ and } h \in H.$$

$$= x_i (hgh)\, h^{-1}\, x_i\, h$$

implies $O\{x g x \mid x \in G\} < \infty \iff O\{h^{-1} x_i h \mid h \in H\} < \infty$

This completes the proof.

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CHAPTER - NINE

SELF COMPRESSED SUBSETS AND SUBGROUPS

1. Introduction : In this chapter, we continue our study of self compression in case of non-abelian groups. We have already seen that the theorems 2.1, 2.2, 2.5, 2.6, 2.14, 2.15, 2.16, 2.17, 2.20 and 2.21 hold for both abelian and non-abelian groups. In several other cases, in contrast to the study of self compression in abelian groups, the situations here are more complicated and results are much more interesting. Besides proving some simple results giving some criteria for self compression of elements, subsets and subgroups, we establishh some results regarding the set of all self compressed elements with respect to a given element. We prove that a necessary and sufficient condition for the set of all self compressed elements with respect to a fixed element to be a subgroup is that the square of the element be identity and show that it is essentially the normalizer of the element (Theorems 9.1 and 9.2). We also prove that an element of odd order can be self compressed only by an element of order 2. This result is although quite simple but seems to be extremely important from the point of view of structural analysis of finite groups. We show that the class of all c.s.c-subgroups form a subclass of normal subgroups. Further we find out some basis results about the subgroup G^* generated by the squares of elements of a group G and characterize a completly self compressed subgroup H by the condition $H \supseteq G^*$ (Theorem 9.16). Lastly, we have defined the concepts of C-measure and invert C-measure to facilitate the study of self compression and have proved some interesting results in this direction.

2. The Set Of All Self Compressed Elements With respect to An Element :

Firstly, we prove a theorem regarding the structure of the set of all self compressed elements of a group with respect to a given element of the group, and give a criterion for such a set to be a subgroup. We also establish that if this set be a subgroup, then it coincides with the normalizer of the given element.

Theorem 9.1 - Let G be a group and $x \in G$, then the set S of all self compressed elements of G with respect to x is a subgroup if, and only if, $x \in S$.

Proof. Let S be a subgroup. Then since $e \in S$

$$\Longrightarrow \quad x^2 = e$$

$$\Longrightarrow \quad x\,xx = x$$

$$\Longrightarrow \quad x \in S$$

Conversely, if $x \in S$, then we have

$$x^2 = e$$

Now if $s_1, s_2, \in S$ be arbitrary, we have

$$x\,s_1\,x = s_1\,, \quad x\,s_2\,x = s_2$$

$$\Longrightarrow \quad s_1\,s_2^{-1} = x\,(s_1\,s_2^{-1})\,x^{-1}$$

$$\Longrightarrow \quad x(s_1\,s_2^{-1})x = s_1\,s_2^{-1} \qquad \text{since } x^{-1} = x$$

$$\Longrightarrow s_1 s_2^{-1} \in S$$

$$\Longrightarrow S \text{ is a subgroup.}$$

Hence the proof is complete.

Cor. 9.1 - For any element x of a group G, the set S of all self compressed elements of G with respect to x is a subgroup if, and only if, $x^2 = e$.

Theorem 9.2 - Let S be the set of all self compressed elements of a group G with respect to an element $x \in G$, then if S be a subgroup, $S = N_G(x)$.

Proof. Since S is a subgroup, we have from Cor. 9.1, that $x^2 = e$. Thus the proof is obvious from the identity:

$$x^{-1} s x = x s x$$

$$= s \qquad \text{where } s \in S.$$

Def. 9.1 - A subset S_1 of a semigroup S with identity is called an ' invert set ' if $s \in S_1$ implies $s^{-1} \in S_1$.

Theorem 9.3 - For any element x of a group G, the set S of all self compressed elements with respect to x, if non-empty, is an invert set and $x^{-1}(s_1 s_2)x = s_1 s_2$ for any $s_1, s_2 \in S$.

Proof. Let $s_1, s_2 \in S$ be arbitrary. We have

$$x . s_1^{-1} x = x(x s_1 x)^{-1} x$$

$$= s_1^{-1}$$

$$\Longrightarrow s_1^{-1} \in S$$

Thus S is an invert set. Finally, we have

$$s_1 s_2 = (x^{-1} s_1 x^{-1})(x\, s_2 x)$$
$$= x^{-1}(s_1\, s_2)x$$

Hence the proof is complete.

3. Order Of An Element And Of Its Self Compressed Element :

We show, below, that if an element of odd order is self compressed with respect to a non-identity element, then that element must be of order two; however, it may not be necessary that the odd order element be self compressed by all elements of order two in the group under consideration. We have no definite result in this direction for elements of even order.

Theorem 9.4 - Let G be a group and $x, g \in G$, such that $O(g)$ is odd, then if $xgx = g$, $x^2 = e$ but not conversely.

Proof. Let $xgx = g$. If $O(g) = 2n+1$ for some natural number n, then

$$(xgx)^{2n+1} = e$$

$$\Longrightarrow (xgx)^{2n}(xgx) = e$$

$$\Longrightarrow (xg)^{2n}(xgx) = e$$

$$\Longrightarrow (xg)(xg)^{2n}x = e$$

$$\Longrightarrow (xg)g^{2n}x = e$$

$$\Longrightarrow xg^{2n+1}x = e$$

$$\Longrightarrow x^2 = e$$

The converse is false :

Let $G = S_3$, the symmetric group of degree 3.

and $g = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, $x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

Here $O(g) = 3$, $O(x) = 2$, but

$$xgx = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \neq g$$

Hence the theorem is complete.

4. Criteria For Self Compression Of An Element :

The following two criteria for self compression of an element with respect to a given element will be of interest.

Theorem 9.5 - For any two elements x, g of a group G, $xgx = g$ if and only if, $xg^ix = g^i$ for every odd integer i.

Proof. Let

$$xgx = g \quad \ldots\ldots\ldots \quad \text{(i)}$$

Consider $x\,g^i x$ for any odd integer $i > 0$. We have from (i)

$$x\,g^i\,x = g\,x^{-1}\,g^{i-1}\,x$$

$$= g^2\,x\,g^{i-2}\,x$$

Continuing in this manner, we get

$$x\,g^i\,x = g^i\,x^{-1}\,x$$

$$= g^i$$

Further if $i < 0$, then since from (i)

$$x^{-1}\,g^{-1}\,x^{-1} = g^{-1}$$

$$\Longrightarrow \quad x\,g^{-1}\,x = g^{-1}$$

We have as above

$$x(g^{-1})^j x = (g^{-1})^j \quad \text{for every odd integer } j > 0$$

$$\Longrightarrow \quad x\,g^i\,x = g^i \qquad \text{for all odd integers } i < 0.$$

Conversely, if the condition be satisfied, the proof is trivial.

This proves the theorem.

Theorem 9.6 - In any group G, for $x,g \in G$, $xgx = g$ if, and only if, $[g,x] = x^2$.

Proof. Let

$$x\,g\,x = g$$

$$\Longrightarrow [g,x] = g^{-1}x^{-1}\,gx$$

$$= g^{-1}(x^{-1}g\,x^{-1})x^2$$

$$= g^{-1}g\,x^2$$

$$= x^2$$

Conversely, if the condition be satisfied, we have

$$[g,x] = x^2$$

$$\Longrightarrow g^{-1}\,x^{-1}\,gx = x^2$$

$$\Longrightarrow g^{-1}\,x^{-1}\,g = x$$

$$\Longrightarrow x\,g\,x = g$$

This completes the proof.

Cor. 9.2 - Let G be a group and $x, g \in G$, such that $O(g)$ is odd, then $x\,gx = g$ implies $[g,x] = e$. However, the converse holds, if $O(x) = 2$.

(The proof follows obviously from theorems 9.4 and 9.6)

5. A Structural Theorem:

Theorem 9.7 - Let G be any group, then

(a) The product of any two c.s.c-subsets of G is a normal subset.

(b) For any two elements $g_1, g_2 \in G$,

(i) If g_1 is c.s.c-element and $g_1 g_2$ is self conjugate $\Longrightarrow g_2$ is c.s.c-element.

(ii) If g_1 and $g_1\, g_2$ are c.s.c-elements $\Longrightarrow g_2$ is self conjugate element.

(iii) If g_1 is self conjugate and $g_1\, g_2$ c.s.c-element $\Longrightarrow g_2$ is c.s.c-element.

Proof. (a) Let S_1, S_2 be any two s.c.s.c-subsets of G. For any $x \in G$, we have

$$x^{-1}(S_1\ S_2)x = (x^{-1}\ S_1\ x^{-1})\ (x\ S_2\ x)$$

$$= S_1\ S_2$$

$\Longrightarrow$ $S_1\ S_2$ is a normal subset.

(b) (i) For any $x \in G$,

$$x^{-1}(g_1 g_2)x = g_1\ g_2$$

$$\Longrightarrow (x^{-1}\ g_1\ x^{-1})(x\ g_2\ x) = g_1\ g_2$$

$$\Longrightarrow g_1(x\ g_2\ x) = g_1\ g_2$$

$$\Longrightarrow x\ g_2\ x = g_2$$

Thus g_2 is c.s.c-element.

(ii) Given any $x \in G$,

$$x(g_1 g_2)x = g_1\ g_2$$

$$\Rightarrow (x\, g_1\, x)(x^{-1}\, g_2\, x) = g_1\, g_2$$

$$\Rightarrow g_1 (x^{-1}\, g_2\, x) = g_1\, g_2$$

$$\Rightarrow x^{-1}\, g_2\, x = g_2$$

Thus g_2 is self conjugate.

(iii) Let $x \in G$ be any element, then

$$x(g_1\, g_2)x = g_1\, g_2$$

$$\Rightarrow (x\, g_1\, x^{-1})(x\, g_2\, x) = g_1\, g_2$$

$$\Rightarrow g_1 (x\, g_2\, x) = g_1\, g_2$$

$$\Rightarrow x\, g_2\, x = g_2$$

Thus g_2 is a c.s.c-element.

Hence the theorem is complete.

Note : The theorem 9.7(b) remains valid even if we replace g_2 by a subset of G . In view of this fact, the following results are interesting.

Cor.9.3 - For any subgroup H of a group G,

(i) If any coset of H corresponding to a c.s.c-element of G be self conjugate, then H is a c.s.c-subgroup.

(ii) If any coset of H corresponding to a c.s.c-element of G be a c.s.c-subset, then H is normal.

(iii) If any coset of H corresponding to a self conjugate element of G be a c.s.c-subset, then H is a c.s.c-subgroup.

6. Some Theorems On Self Compression Of Subsets:

Theorem 9.8 - Let K be a subset of a group G, then the set $\overline{K}$ of all self compressed subsets of G with respect to K whose normalisers contain K is a semigroup with respect to set product.

(The proof of the theorem is straight forward)

Theorem 9.9 - If any subset S of a group G is both self conjugate and self compressed with respect to $x \in G$, then the subgroup generated by S is self compressed with respect to x but the converse need not hold.

Proof. (It immediatly follows in view of theorem 2.7)

The behaviour of self compressed subsets under fundamental mappings and direct products follows immediatly from the corresponding results for abelian groups, hence we pass on to the following:

7. Self Compression of Subgroups:

We prove, below, that the class of all c.s.c-subgroups of a group form a subclass of normal subgroups of the group and establish two criteria for complete self compression of a subgroup in terms of its cosets. In the end, we prove a theorem regarding the self compression of all non-identity subgroups of a periodic group.

Theorem 9.10 - If a subgroup H of a group G be self compressed with respect to an element in G, then it is self conjugate with respect to the same element but not conversely.

Proof. Let H be self compressed with respect to $x \in G$, then

$$x\,H\,x = H$$

$$\Longrightarrow \quad x\,e\,x = x^2 \in H$$

Hence

$$x^{-1}H\,x = x^{-1}\,x\,H\,x\,x$$

$$= H\,x^2$$

$$= H$$

The falsity of the converse is evident, since if we take $H = e$, then H is normal but not necessarily completly self compressed.

Hence the theorem is complete.

Cor.9.4 - Every c.s.c-subgroup of a group is normal but the converse need not hold.

Theorem 9.11- A subgroup H of a group G is self compressed with respect to a subset S of G, if, and only if, every coset $x\,H (\neq H)$ for any $x \in S$ is of order 2.

Proof. Let H be self compressed with respect to S, then for any coset $x\,H\,(\neq H)$ with respect to $x \in S$

$$(x\,H)^2 = (xH)(xH)$$

$$= (xH)\,(H\,x) \qquad \text{(Theorem 9.10)}$$

$$= xHx$$

$$= H$$

Conversely, if the condition be satisfied, the for. any $x \in S$

$$(xH)^2 = H$$

$$\Longrightarrow xHx \subseteq H$$

$$\Longrightarrow x^{-1}Hx^{-1} = xHx \subseteq H$$

$$\Longrightarrow H \subseteq xHx$$

Hence

$$H = xHx$$

This completes the proof.

Cor.9.5 - A subgroup H of a group G is completly self compressed if, and only if, every coset $xH (\neq H)$ of H in G is of order 2.

Theorem 9.12- A subgroup H of a group G is completly self compressed if, and only if, for $x, y \in G$, $xH = y^{-1}H$ implies $Hx = Hy$.

Proof. If H be a c.s.c-subgroup, then the proof is trivial because of normality of H. Conversely, let the condition be satisfied.

Now for any $x \in G$, $xhH = (x^{-1})^{-1}H$ where $h \in H$

$$\Longrightarrow Hxh = Hx^{-1}$$

$$\Longrightarrow \quad Hxhx = H$$

$$\Longrightarrow \quad xhx \in H$$

$$\Longrightarrow \quad xHx \subseteq H$$

Similarily, taking x^{-1} for x, we have

$$x^{-1} H x^{-1} \subseteq H$$

$$\Longrightarrow \quad H \subseteq xHx$$

Hence

$$H = xHx$$

This proves the theorem.

Theorem 9.13 - If every subgroup $H(\neq e)$ of a periodic group G is self compressed with respect to a subset K of G, then $yxy = x^i$, $i \in I$, for every $y \in K$ and any $x(\neq e) \in G$, however, the converse holds if and only if $N_G(H) \supseteq K$.

Proof. The proof is obvious in view of theorem 2.4 .

8. Properties Of Subgroup Generated By Squares of Elements Of A Group:

We denote the subgroup generated by the squares of elements of a group G by G^* and study its properties. The properties obtained here throw immense light upon the structure of G^* in non-abelian groups.

Def. 9.2 - For any group G, we define G^* to be the subgroup of G generated by squares of elements of G.

We may note here that if G be an abelian group, then $G^* = \{g^2 \mid g \in G\}$, hence the concept of G^* introduced in chapter three is essentially the same as given in the definition 9.2. The following properties cam be easily verified.

(i) The subgroup G^* is fully invariant (For any endomorphism η of G, $(g^2)\eta = (g\eta)^2$ for every $g \in G$).

(ii) The subgroup generated by the squares of elements of a subset S is contained in any subgroup H self compressed with respect to S, however, the converse is not true in general.

(iii) If $N_G(H) \supseteq S$, the converse of (ii) holds.

(iv) G^* is a c.s.c-subgroup of G (Follows from (i), (ii) and (iii).

(v) Every normal subgroup of G containing G^* is a c.s.c-subgroup.

(vi) For any group G, $G^* \supseteq G'$, the commutator subgroup of G (This is immediate in view of Cor. 9.5).

(vii) The subgroup G^* of a group G is torsion group if, and only if, the subgroup H^* of every finitely generated subgroup H of G has this property.

(viii) If the subgroup G^* of a group G is torsion group, then G is torsion.

Theorem 9.14 - For any group G, $G^* \subseteq Z(G)$ implies G^* is abelian but not conversely.

Proof. If $G^* \subseteq Z(G)$, then clearly G^* is abelian. To prove that the converse is not true in general, let

$$G = S_3, \text{ the symmetric group of degree 3.}$$

Then

$$G^* = \left\{ e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \alpha^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \right\}$$

$$= [\alpha], \text{ the cyclic group generated by } \alpha.$$

and

$$Z(G) = e$$

Thus G^* is abelian but $G^* \not\subseteq Z(G)$.

This completes the proof.

Cor.9.6 - If a group G is abelian then its subgroup G^* is abelian but the converse nedd not hold.

Reamrk : We observe that for a group G and for any subgroup H of G,

(i) $[Z(G)]^* \neq Z(G^*)$ (ii) $N_G(H^*) \neq [N_G(H)]^*$ in genreral . The following example emplains the situations.

Consider $G = S_3$, the symmetric group of degree 3,

and $H = G$.

Then $G^* = [\alpha]$, the cyclic group generated by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

and $$Z(G) = e$$

But we see that in this case

$$G^* = Z(G^*) \supset [Z(G)]^* = e$$

and

$$N_G(H^*) = G \supset [N_G(H)]^* = G^*$$

Theorem 9.15 - For any subgroup H of a group G, $N_G(H) \subseteq N_G(H^*)$.

Proof. We know that every element of H^* is of the type $h_1^2 . h_2^2 \ldots\ldots h_n^2$ where h_i's are in H.

Now for any $g \in N_G(H)$,

$$g^{-1}(h_1^2 . h_2^2 \ldots h_n^2)\, g = (g^{-1}h_1 g)^2 (g^{-1}h_2\, g)^2 \ldots (g^{-1}h_n g)^2$$

$$= h_1'^2 . h_2'^2 \ldots h_n'^2 \text{ where } h_i' \in H$$

$$\Longrightarrow g^{-1} H^* g \subseteq H^*$$

Further, since g^{-1} also belongs to $N_G(H)$, we have as above

$$g H^* g^{-1} \subseteq H^*$$

$$\Longrightarrow H^* \subseteq g^{-1} H^* g$$

Hence

$$H^* = g^{-1} H^* g$$

$$\Longrightarrow g \in N_G(H^*)$$

$$\Longrightarrow N_G(H) \subseteq N_G(H^*)$$

This proves the theorem completly.

8. G^* And Self Compression:

In what follows, we prove results, which display that the role played by G^* in self compression in non-abelian groups is almost similar to the role it played in abelian groups.

Theorem 9.16 - A subgroup H of a group G is completly self compressed if, and only if, $H \supseteq G^*$.

Proof. Necessity is obvious. To prove the sufficient part, let $x \in G$ and $h \in H$ be arbitrary. Since G^* is a c.s.c-subgroup of G, it follows from Cor.9.5 that G/G^* is an abelian group, hence

$$(x G^*)\ (h G^*) = (h G^*)(x G^*)$$

$$\Longrightarrow\ x h G^* = h x G^*$$

$$\Longrightarrow\ x(h G^*)x = h(x G^* x)$$

$$= h G^*$$

$$\Longrightarrow\ x H x = H \quad \text{for all } x \in G$$

Hence the proof is complete.

Remark : The above theorem suggests that G^* is the smallest c.s.c-subgroup of G. Further, it is easy to see, in view of Cor.9.4 and theorems 9.11 and 9.16, that the theorems 7.1, 7.2, 7.3, 7.5, 7.6, 7.7, 7.8 and 7.12 on compression series and chains hold in non-abelian groups as well.

Theorem 9.17 - A subgroup H of a group G is completly self compressed if, and only if, H contains a normal c.s.c-subset $\mathcal{S}$ of G.

(Proof follows immediatly from theorems 9.9, 9.16 and Cor.9.4).

10. Invert C-Measure A nd C-Measure:

Def. 9.3 - Let G be a group. To every pair of elements $g, x \in G$, we associate an element.

$$[g,x]^* = g.\ g_x$$

We call $[g,x]^*$ the invert C-measure of the pair (g,x).

The following properties of invert C-measure are trivial.

(i) $[g,x]^* \ [x^{-1}, g^{-1}]^* = e$, hence $([g,x]^*)^{-1} = [x^{-1}, g^{-1}]^*$

(ii) $[g,x]^* = e$ if, and only if, $[x,g]^* = e$.

(iii) $[x,x^{-1}]^* = e$

(iv) $[g,x]^* = e$ if and only if $g_x = g^{-1}$.

(v) $[g,x]^* = g^2$ if and only if $g_x = g$.

(vi) $[g,x]^* = x^{-1}[x,g]^* x = g[x,g]^* g^{-1}$

(vii) $[g,x]^* [x,g] = x_g^2$

(viii) $[[g,x]^*, x^{-1}]^* = [gx , g]^* = [g,x g]^*$

(ix) $[x_1^{-1} g x_1, x_1^{-1} x x_1]^* = x_1^{-1}[g,x]^* x_1$ for every $x_1 \in G$.

(x) $[g\,f,\,x]^* = ([f,x]^*)_g\,[g,x]\,x^{-1}\,[g,f]\,x$

We can regard the formation of invert C-measure as an operation defined over the set of group elements. This operation is not in general associative, that is if $g_1, g_2, g_3 \in G$, then

$$\left[[g_1,\,g_2]^*,\,g_3\right]^* = \left[g_1,\,[g_2,\,g_3]^*\right]^*$$

does not always hold. The associativity does not hold even in case of abelian groups, however, the following result holds.

Theorem 9.18 - The operation of formation of invert C-measure is associative in a group G if, and only if, every element ($\neq e$) of G is of order 2.

Proof. Let the operation of formation of invert C-measure be associative in G, then if $g \in G$

$$\left[[g,\,e]^*,\,e\right]^* = \left[g,\,[e,e]^*\right]^*$$

$$\Longrightarrow [g^2,e]^* = [g,e]^*$$

$$\Longrightarrow g^4 = g^2$$

$$\Longrightarrow g^2 = e$$

Thus every element ($\neq e$) of G is of order 2. Conversely, if the condition be satisfied, then G is an abelian group with elements of order 2, hence for any $g_1, g_2, g_3 \in G$

$$\left[[g_1,\,g_2]^*,\,g_3\right]^* = \left[g_1,\,[g_2,g_3]^*\right]^*$$

This completes the proof.

We now prove some results about c.s.c-subgroups in a group with the help of the notion of invert C-measure.

Theorem 9.19 - H is a c.s.c-subgroup of a group G if, and only if, $h \in H$, $x \in G$ implies $[h,x]^* \in H$.

Proof. Let the condition be satisfied, then if $h \in H$ and $x \in G$ be arbitrary , we have

$$[h, x]^* = h \cdot h_x \in H$$

$$\Longrightarrow \quad H_x \subseteq H$$

Similarily, putting x^{-1} for x, we have

$$H_{x^{-1}} \subseteq H$$

$$\Longrightarrow \quad H \subseteq H_x$$

Hence

$$H = H_x$$

Thus H is c.s.c-subgroup. Conversely, if H be a c.s.c-subgroup, then the proof is trivial.

Hence the theorem is complete.

We know that the product of two invert C-measures of a group G need not be an invert C-measure, hence they do not form a group in general, however, the following result holds.

Theorem 9.20 - Let G(I C) denotes the subgroup generated by all invert C-measures of a group G, then $G(I\ C) = G^*$.

Proof. Let $g \in G$ be arbitrary, then

$$[g,e]^* = g^2 \in G(I\ C)$$

$$\Longrightarrow \quad G^* \subseteq G(I\ C)$$

Conversely, if $g_1, g_2 \in G$, then

$$[g_1, g_2]^* = (g_1 g_2)^2 \in G^*$$

$$\Longrightarrow G(I\ C) \subseteq G^*$$

Hence

$$G(I\ C) = G^*$$

This proves the lemma completly.

We, now, give a proof of a well known result giving $G' \subseteq G^*$ for any group G, with the help of the notion of invert C-measure.

Proof. Let $g, g' \in G$ be arbitrary, then from theorem 9.20

$$\begin{aligned} g.g'\ G^* &= (g\ g')^{-1} [g, g']^*\ G^* \\ &= g'^{-1}\ g^{-1}\ G^* \\ &= g'^{-1}\ G^*\ g^{-1}\ G^* \quad (G^* \text{ is normal in } G). \\ &= g'\ G^*\ g\ G^* \\ &= g'g\ G^* \end{aligned}$$

$$\Longrightarrow [g, g']\ G^* = G^*$$

$$\Longrightarrow \quad G' \subseteq G^*$$

Thus the assertion follows.

Cro.9.7 - For any group, $G' = G^*$ if, and only if, G' is a c.s.c-subgroup of G.

Theorem 9.21 - If the subgroup G^* of a group G be finite, then the set $\overline{g}$ of all compressed transforms of any element $g \in G$ is finite.

Proof. For any $x \in G$, we have

$$g_x = g^{-1} [g,x]^*$$

$$\Longrightarrow \quad O(\overline{g}) < \infty \qquad \text{(Theorem 9.20)}$$

Thus the proof is complete.

Remark (i) : We may note that the above theorem holds true even if we suppose the number of invert C-measures in G to be finite, or that the squares of elements of G form a finite set. Thus, in view of theorem 8.12, the following result holds.

Cor.9.8 - Let in a group G, $G_1 = \{g^2 \mid g \in G\}$ is finite, then every conjugate class of elements in G is finite.

Remark (ii) : In view of the remark (i), above and Cor.9.8, it follows that the subgroup H of theorem 8.11 is identical to G itself.

Def.9.4 - Given a group G, to every pair of elements $g, x \in G$ we associate an element defined as

$$\overline{[g,x]} = g^{-1}.g_x$$

We call $\overline{[g,x]}$ the C-measure of (g,x) in the natural way since the C-measure of (g,x) is identity (i.e. $\overline{[g,x]} = e$) if, and only if, $g_x = g$. Thus in certain sense, it is a measure of self compression of g with respect to x. The following properties of C-measure are trivial.

(i) $(\overline{[g,x]})^{-1} = x^{-1}\ \overline{[g,x^{-1}]}\ x$

(ii) $\overline{[g,x]} = ([x,g])_x = ([x,g]^*)_{g^{-1}}$

(iii) $\overline{[g^{-1},x^{-1}]} = g(\overline{[g,x]})^{-1}\ g^{-1}$

(iv) $\overline{[gh,x]} = (\overline{[g,x]}\ \overline{[x,h]})_{h^{-1}}$ for all $x \in G$

(v) $\overline{[g,hx]} = \overline{[g,h]}\ (\overline{[g,x]}\ \overline{[x,h]})_{h^{-1}}$

As in the case of invert C-measure, the operation of formation of C-measure over elements of a group G is not in general associative i.e. for g_1, g_2, $g_3 \in G$

$$\overline{[\overline{[g_1,g_2]},\ g_3]} \neq \overline{[g_1,\overline{[g_2,g_3]}]}$$

in general. This is not true even in the case of abelian groups, however, a result similar to theorem 9.18 holds here as well:

Theorem 9.22 - The operation of formation of C-measure is associative in a group G, if, and only if, every element ($\neq e$) is of order 2.

Proof. Let the operation of formation of C.measure be associative in G, then if $g \in G$, we have

$$\overline{[\overline{[e,e]},\ g]} = \overline{[e,\ \overline{[e,g]}]}$$

$$\Longrightarrow \quad g^2 = g^4$$

$$\Longrightarrow \quad g^2 = e$$

Thus every element ($\neq e$) of G is of order 2. Conversely, if the condition be satisfied, then G is an abelian group with elements of order 2, hence if $g_1, g_2, g_3 \in G$

$$\overline{[\overline{[g_1,g_2]},\ g_3]} = \overline{[g_1^{-1}g_2\ g_1\ g_2,\ g_3]}$$

$$= \overline{[e,\ g_3]} = e$$

Also

$$\overline{[g_1, \overline{[g_2,g_3]}]} = e$$

Thus the proof is complete.

Cor.9;9 - C-measure is an associative operation over elements of a group G if, and only if, invert C-measure is associative over elements of G.

Theorem 9.23 - Let H be a subgroup of a group G and $h \in H$, $x \in G$, then $\overline{[h,x]} \in H$ if, and only if, $[h,x]^* \in H$.

Proof. The proof follows immediatly from the identity:

$$[h,x]^* = h^2\widehat{[h,x]}$$

Theorem 9.24 - H is a c.s.c-subgroup of group G if and only if $h \in H$, $x \in G$ implies $\overline{[h,x]} \in H$.

Proof. The proof is obvious from theorems 9.23 and 9.19.

Theorem 9.25 - For any group G, the subgroup G(C) of G generated by all C-measures of G equals to G^*.

Proof. The proof is obvious in view the theorems 9.20 and 9.23.

Remark : The theorem 9.21 and the result that $G' \subseteq G^*$ can also be easily proved with the help of the notion of C-measure in a group. The behaviour of both these concepts is almost parallel and any one of these can be used according to requirements.

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CHAPTER - TEN

COMPRESSOR IN NON - ABELIAN GROUPS

1. Introduction : The study of compressor in abelian groups, as we mentioned in the main introduction, has been attempted to fulfill the deficiency created by triviality of the concept of normality in abelian case. This concept is, however, quite interesting even in case of non-abelian groups. The study, here in this dissertation, is not made exhaustively, even then some of the interesting aspects have been observed. We note that in non-abelian groups, the compressor of a subgroup is not a subgroup in general and therefore find a criterion for the same to be a subgroup (Theorem 10.1). Further, besides proving some simple results inter relating compressors and normalisers of subsets and subgroups(§ 3 and § 4), we establish : 'The compressor of a subgroup is a normal subset of its normalizer (Theorem 10.3)', 'The normaliser of the compressor of a sylow π - subgroup of a group is the compressor of the normaliser of the subgroup, moreover, both are equal to the normalizer of the subgroup (Theorem 10.5) and 'If a subgroup H_1 contains the normaliser of a sylow π - subgroup H and all the sylow π - subgroups of H_1 be conjugate, then the compressor of H_1 is H_1 itself' etc. Finally, we prove some basic results about compressors of elements and subgroups and put some important remarks in this direction.

2. Structure Of Compressor :

From the theorem 4.3, the compressor of any subset of an abelian group is a subgroup, but, evidently the compressor of a subset (or subgroup) of a group in general is not a subgroup. We start with finding out a criterion for the compressor of a subgroup of a group to be subgroup, and further prove that, however, the compressor of a subgroup is not necessarily a subgroup but every coset of the subgroup with respect to an element of the compressor is contained in it.

Theorem 10.1 - Let H be a subgroup of a group G, then $C_G(H)$ is a subgroup of G if and only if $x,y \in C_G(H)$ implies $[x,y] \in H$.

Proof. Suppose $C_G(H)$ be a subgroup of G then for any $x,y \in C_G(H)$

$$xy\,H\,xy = H$$

$$\Longrightarrow\ x^{-1}y^{-1}xy\,H = x^{-1}y^{-1}\,H\,y^{-1}x^{-1}$$

$$\Longrightarrow\ [x,y]\,H = H$$

$$\Longrightarrow\ [x,y] \in H$$

Conversely, if the condition be satisfied, then let $x,y \in C_G(H)$

$$xHx = H$$

$$\Longrightarrow\ x^{-1}H\,x^{-1} = H$$

Thus $x \in C_G(H) \Longrightarrow x^{-1} \in C_G(H)$

Also

$$xy\,H\,xy = xy\,H\,yy^{-1}xy$$

$$= xH\,y^{-1}xy$$

$$= x\,Hx\,[x,y]$$

$$= H\cdot[x,y]$$

$$= H \quad \text{since} \quad [x,y] \in H$$

$$\Longrightarrow \; xy \in C_G(H)$$

Hence $C_G(H)$ is a subgroup.

This proves the theorem.

Theorem 10.2 - For any subgroup H of a group G, $x \in C_G(H)$ implies $xH \subseteq C_G(H)$.

Proof. (The proof is obvious)

Cor. 10.1 - The compressor $C_G(H)$ of a subgroup H of a group G is a union of cosets of H with respect to elements of $C_G(H)$.

3. Compressors And Normalizers Of Subgroups:

This section is devoted to the study of relations between compressors and normalizers of subgroups. We prove that the compressor of a subgroup is a normal subset of its normaliser.

Besides proving some simple results in this connection, we also establish that the compressor of the normalizer of a sylow π - subgroup of a group equals to the normalizer of the compressor of the subgroup and both these coincide to the normalizer of the subgroup.

Theorem 10.3 - For any subgroup H of a group G, $C_G(H)$ is a normal subset of the normalizer $N_G(H)$ of H in G.

Proof. Evidently, theorem 9.10

$$\implies C_G(H) \subseteq N_G(H)$$

Now let $x \in N_G(H)$ be arbitrary, then by cor 10.1, we have

$$x^{-1} C_G(H)x = x^{-1}(U\, x_i H)x \quad \text{where } x_i\text{'s} \in C_G(H)$$

$$= U(x^{-1}x_i x)H \quad \text{since } x \in N_G(H)$$

Also

$$x^{-1}x_i x(H)\, x^{-1}\, x_i\, x = x^{-1}x_i(x H x^{-1})\, x_i x$$

$$= x^{-1}(x_i H x_i)x$$

$$= x^{-1} H x \quad \text{since } x_i \in C_G(H)$$

$$= H$$

$$\implies x^{-1}\, x_i\, x \in C_G(H)$$

Hence

$$x^{-1} C_G(H)x \subseteq C_G(H) \qquad \text{(Theorem 10.2)}$$

$\Longrightarrow x^{-1}\, C_G(H)x = C_G(H)$ since x^{-1} is also in $N_G(H)$.

$\Longrightarrow C_G(H)$ is a normal subset of $N_G(H)$.

This proves the theorem.

Cor. 10.2 - For any subgroup H of a group G,

$$N_G(H) \subseteq N_G(C_G(H)).$$

Theorem 10.4 - Let H be a subgroup of a group G, then H is normal in $C_G(H)$, whereas $C_G(H)$ is the maximal set in which H is complètly self compressed.

Proof. (First part of the theorem follows from the theorem 9.10 and rest is obvious inview of definition 4.1).

Evidently, for any subgroup H of an abelian group G, $N_G(C_G(H)) = C_G(N_G(H)) = N_G(H)$, but, it is not true in general, however, the following theorem holds:

Theorem 10-5 - Let H be a sylow π - subgroup of a group G, then

$$N_G(C_G(H)) = C_G(N_G(H)) = N_G(H)$$

Proof. We know that every sylow π -subgroup of a group is the only sylow π - subgroup of its normaliser. For $x \notin N_G(H)$

$$x^{-1} H x \neq H$$

Now, if

$$x^{-1} C_G(H)\, x \subseteq C_G(H)$$

$$\Longrightarrow x^{-1} H x \subseteq C_G(H)$$

$$\Longrightarrow x^{-1} H x \subseteq N_G(H)$$

A contradiction that H is the only sylow π - subgroup of its normalizer, hence

$$x^{-1} C_G(H) x \not\subseteq C_G(H)$$

$$\Longrightarrow N_G(C_G(H)) \subseteq N_G(H)$$

$$\Longrightarrow N_G(C_G(H)) \subseteq C_G(N_G(H)) \quad \text{— (i)}$$

Further, we know that

$$N_G(N_G(H)) = N_G(H)$$

and

$$C_G(N_G(H)) \subseteq N_G(N_G(H))$$

$$\Longrightarrow C_G(N_G(H)) \subseteq N_G(H) \quad \text{--- (ii)}$$

Hence from (i), (ii) and Cor. 10.2,

$$N_G(H) \subseteq N_G(C_G(H)) \subseteq C_G(N_G(H)) \subseteq N_G(H)$$

$$\Longrightarrow N_G(C_G(H)) = C_G(N_G(H)) = N_G(H)$$

This completes the proof.

Theorem 10.6 - Let H be a subgroup of a group G such that $N_G(H) = H$, then $C_G(H) = H$ but not conversely.

Proof. Evidently

$$H \subseteq C_G(H) \subseteq N_G(H) = H$$

$$\implies C_G(H) = H$$

To rpove the falsity of the converse, consider

G, a non identity group with no element of order 2.

$$H = e$$

Then

$$C_G(H) = H \quad \text{but} \quad N_G(H) = G$$

This completes the theorem.

Theorem 10.7 - Let H be a sylow π - subgroup of a group G, and H_1 a subgroup of G containing $N_G(H)$, then if all sylow π- subgroups of H_1 be conjugate, we have

$$C_G(H_1) = H_1$$

Proof. Evidently, for such a subgroup H_1

$$N_G(H_1) = H_1$$

$$\implies C_G(H_1) = H_1 \qquad \text{(Theorem 10.6)}$$

Hence the proof is complete.

4. Compressors And Normalisers Of Elements :

The results, in this section give the mutual relationships between compressors and normalisers of elements in different circumstances.

Theorem 10.8 - Let G be a group and $g, g' \in G$, such that $g' = g\, g_1$ for some $g_1 \in G$, then $x \in C_G(g)$ implies $x \in C_G(g')$ if, and only if, $x \in N_G(g_1)$.

Proof. Let for $x \in C_G(g)$, $x \in C_G(g')$. Consider

$$\begin{aligned} g' &= x\, g' x \\ &= x\, g\, g_1 x \\ &= x\, g x.\ x^{-1} g_1\, x \\ &= g . x^{-1} g_1\, x \end{aligned}$$

$$\begin{aligned} &\Longrightarrow\ g\, g_1 = g . x^{-1} g_1 x \\ &\Longrightarrow\ x^{-1} g_1 x = g_1 \\ &\Longrightarrow\ x \in N_G(g_1)\ . \end{aligned}$$

Conversely, if the condition be satisfied, then

$$\begin{aligned} x\, g' x &= x\, g g_1 x \\ &= x\, g\, x .\ x^{-1} g_1 x \\ &= g\, g_1 \quad \text{since } x \in N_G(g_1),\ x \in C_G(g) \end{aligned}$$

$$\begin{aligned} &\Longrightarrow\ x\, g' x = g' \\ &\Longrightarrow\ x \in C_G(g') \end{aligned}$$

Thus the proof is complete.

Theorem 10-9 - Let G be a group and $g_1, g_2 \in G$, then

(i) $x \in C_G(g_1) \cap N_G(g_2) \implies x \in C_G(g_1 g_2)$

(ii) $x \in C_G(g_1) \cap C_G(g_2) \implies x \in N_G(g_1 g_2)$

(iii) $x \in C_G(g_1) \cap N_G(g_1) \implies x^2 = e.$

Proof. (i) By given

$$x g_1 x = g_1, \qquad x^{-1} g_2 x = g_2$$

$$\implies x g_1 g_2 x = x g_1 x x^{-1} g_2 x$$

$$= g_1 g_2$$

$$\implies x \in C_G(g_1 g_2) .$$

(ii) We have

$$x g_1 x = g_1, \quad x g_2 x = g_2$$

$$\implies x g_1 g_2 x = x g_1 x \cdot x^{-1} g_2 x$$

$$= g_1 \cdot x^{-1} g_2 x^{-1} \cdot x^2$$

$$= g_1 g_2 x^2$$

$$\implies g_1 g_2 = x^{-1} g_1 g_2 x$$

$$\implies x \in N_G(g_1 g_2)$$

(iii) We have, by given that

$$x g_1 x = g_1 = x^{-1} g_1 x$$

$$\implies g_1 = g_1 x^2$$

$$\Longrightarrow x^2 = e$$

This completes the theorem.

Note : The results (i) and (ii) of the above theorem hold true even if we replace g_1, g_2 by subsets S_1, S_2 in G.

5. Conjugate Of Compressor :

Here, below, we show that any conjugate of the compressor of a subset is the compressor of the conjugate subset with respect to the same element, and thereby, we deduce a generalisation of Cor. 10.2 .

Theorem 10.10 - Let S be a subset of a group G and $g \in G$, then

$$C_G(g^{-1}S\,g) = g^{-1}\,C_G(S)\,g.$$

Proof. Let $x \in C_G(S)$ be arbitrary. Consider

$$g^{-1}x\,g.\;g^{-1}\,S\,g.\;g^{-1}x\,g = g^{-1}\,x\,S\,x\,g$$

$$= g^{-1}\,S\,g$$

$$\Longrightarrow g^{-1}x\,g \in C_G(g^{-1}S\,g)$$

$$\Longrightarrow g^{-1}C_G(S)g \subseteq C_G(g^{-1}\,S\,g)$$

Conversely, if for $g' \in G$

$$g'.\;g^{-1}S\,g.g' = g^{-1}S\,g$$

$$\Longrightarrow g\,g'g^{-1}.\ S.\ g\,g'g^{-1} = S$$

$$\Longrightarrow \quad g\,g'g^{-1} \in C_G(S)$$

$$\Longrightarrow \quad g' \in g^{-1}C_G(S)\,g$$

$$\Longrightarrow \quad C_G(g^{-1}S\,g) \subseteq g^{-1}\,C_G(S)\,g$$

Hence

$$C_G(g^{-1}S\,g) = g^{-1}C_G(S)\,g$$

This completes the proof.

Cor. 10.3 - For any subset S of a group G, $N_G(S) \subseteq N_G(C_G(S))$.

6. Properties And Characterisations Of Compressors:

In what follows, we deal with rather simple properties of compressors of elements and subgroups.

Theorem 10-11 - Let G be a group and $g \in G$, then

(i) $x \in C_G(g)$ if, and only if, $x \in C_G(g^i)$ for every odd integer i.

(ii) $x \in C_G(g)$ if, and only if, $[g,x] = x^2$

(iii) If $O(g)$ = odd, then $x\ (\neq e) \in C_G(g)$ implies $O(x) = 2$

Proof. The proof follows immediately in view of theorems 9.4, 9.5 and 9.6 .

Cor. 10.4 - Let G be a group and $g \in G$, then $C_G(g) = \bigcap_{i \in I'} C_G(g^i)$

where I' denotes the set of all odd integers.

Theorem 10-12 - For any group G, the set $C(G) = \bigcap_{g \in G} C_G(g)$ is is a subgroup of elements of order 2.

Proof. Evidently, every element ($\neq e$) of C(G) is of order 2. Let $x, y \in C(G)$ and $g \in G$ be arbitrary, then

$$x\, g\, x = g$$

$$\Longrightarrow x^{-1} g\, x^{-1} = g$$

$$\Longrightarrow x^{-1} \in C(G)$$

Further, since

$$x\, y = y^2\, x\, y$$

$$= y\, x \qquad (y \in C(G))$$

$$\Longrightarrow x\, y\, g\, xy = x\, y\, g\, yx$$

$$= g$$

$$\Longrightarrow x\, y \in C(G)$$

Hence C(G) is a subgroup of elements of order 2.

This completes the theorem.

Theorem 10-13 - Let H be a subgroup of a group G, then $x \in C_G(H)$ if, and only if, $(x\, H)^2 = H$.

Proof. The proof is obvious in view of theorem 9.11 .

Theorem 10.14 - Let G be a group containing no element of order 2, then for every direct factor G_1 of G, $C_G(G_1) = G_1$.

Proof. Let

$$G = G_1 \times G_2$$

Now, if $g \in G$, $g \notin G_1$, such that $g = g_1 g_2$ where $g_i \in G_i$, $i = 1,2$, then

$$g\ G_1\ g = G_1$$

$$\Longrightarrow \quad g_1 g_2 G_1\ g_1 g_2 = G_1$$

$$\Longrightarrow \quad g_2\ G_1\ g_2 = G_1$$

$$\Longrightarrow \quad g_2^2\ (\neq e) \in G_1 \cap G_2$$

A contradiction that $G = G_1 \times G_2$, hence

$$C_G(G_1) = G_1$$

This proves the theorem.

Remark : We can easily see that if H_1 be a normal subgroup of a group G, then for every subgroup H of G, $C_G(H) \subseteq C_G(H H_1)$ Fruther, we note that if H_1, H_2 be two subgroups of G such that $H_1 \subseteq H_2$, then $C_G(H_1) \nsubseteq C_G(H_2)$ in general, and thus the theorem 4.8 is not true. Ror example:

Let $G = S_3$, the symmetric group of degree 3.

and $H_1 = e$, $H_2 = \{e, (23)\}$

Then, it is easy to see that

$$C_G(H_1) = \{e, (23), (12), (13)\}$$

and $\qquad C_G(H_2) = \{e, (23)\}$

Thus

$$H_1 \subset H_2 \text{ but } C_G(H_1) \not\supseteq C_G(H_2).$$

Note : The theorem 4.10 for compressors remain valid in case of non-abelian groups in view of definition 10.1. Several other interesting aspects of the concept are yet to probed, which have not been done for fear of bulk.

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Appendix

Separatum

PUBLICATIONES MATHEMATICAE

TOMUS 12.
FASC. 1—4.

DEBRECEN
1965

P. K. Sharma

A note on translative mappings

INSTITUTUM MATHEMATICUM UNIVERSITATIS DEBRECENIENSIS
HUNGARIA

PUBLICATIONES MATHEMATICAE

12. KÖTET
1—4. FÜZET

DEBRECEN
1965

A DEBRECENI TUDOMÁNYEGYETEM MATEMATIKAI INTÉZETE

A note on translative mappings*)

By P. K SHARMA (Aligarh)

In [1] M. HOSSZÚ has defined that a system of mappings $F_u: x \to F_u(x)$ of a set X into itself, where u ranges over elements of a set U is called *commutable* if

$$\forall u, v \in U, x \in X, F_u F_v(x) = F_v F_u(x) \tag{1}$$

holds. A system of mappings $F_u: x \to F_u(x)$ is *transitive* if

$$\forall x \in X, F_U(x) = X$$

holds, where the set of elements $F_u(x)$ $(u \in U)$ is denoted by $F_U(x)$. We, in this paper, define that if X is a groupoid with respect to operation of multiplication then a system of mappings $F_u: x \to F_u(x)$ of X into X is called *right translative* or *left translative* or *translative* respectively according as

$$\forall x, y \in X, u \in U \tag{2}$$

$$F_u(xy) = x F_u(y)$$

or

$$F_u(xy) = F_u(x) y$$

or

$$F_u(xy) = F_u(x) y = x F_u(y)$$

holds.

Theorem 1. *A given mapping F_u of a translative system of mappings over a semigroup X is $1-1$ if the (rt. or lt.) cancellation law holds with respect to every element of $F_u(X)$.*

PROOF. We shall prove the theorem supposing that the right cancellation law holds w. r. to every element of $F_u(X)$. Let for $x, y \in X$

$$F_u(x) = F_u(y).$$

Then

$$F_u(xy) = F_u(x) y = F_u(y) y = F_u(y^2).$$

Also

$$F_u(xy) = x F_u(y) = x F_u(x) = F_u(x^2).$$

*) This research was financially supported by the Council of Scientific Research and Industrial Affairs, Govt. of India.

Hence,

$$F_u(x^2)=F_u(y^2)\Rightarrow xF_u(x)=yF_u(y)\Rightarrow xF_u(x)=yF_u(y)\Rightarrow x=y$$

This completes the proof.

Theorem 2. *If X is a semigroup such that $yX=X$ for some $y\in X$ then:*

(a) *Every mapping F_u of a transitive system of translative mappings over X maps X onto itself.*

(b) *Every translative mapping F_u of X into X is $1-1$ if left cancellation w. r. to $F_u(y)$ holds.*

PROOF. (a) We have

$$F_u(X)=F_u(F_v(x))=F_u(y\,F_v(x))=F_u(F_v(y)\,x)=$$
$$=F_v(y)\,F_u(x)=F_v(y\,F_u(x))=X.$$

(b) Let $x_1, y_1\in X$. We have

$$F_u(x_1)=F_u(y_1)\Rightarrow F_u(yx')=F_u(yx'')\text{ where } x', x''\in X\Rightarrow$$
$$\Rightarrow F_u(y)\,x'=F_u(y)\,x''\Rightarrow x'=x''\Rightarrow yx'=yx''\Rightarrow x_1=y_1.$$

Hence the proof is complete.

Theorem 3. *If X is a semigroup with identity e then:*

(a) *Every system of translative mappings over X is commutable.*

(b) *If there exists a transitive system of translative mappings over X then X i abelian.*

(c) *The system of all translative mappings over X is an abelian semigroup. If there exists a translative mapping F_{u_0} such that $F_{u_0}(e)=e$ then it coincides with the identity mapping.*

PROOF. (a) $\forall x\in X,\ u, v\in U$,

$$F_uF_v(x)=F_uF_v(ex)=F_u(F_v(e)\,x)=$$
$$=F_v(e)F_u(x)=F_v(e\,F_u(x))=F_vF_u(x).$$

(b) Let $x, y\in X$ be arbitrary, then if $u\in U$ s. t.

$$F_u(e)=y$$

(such a u always exists by supposition of transitivity), then we have

$$F_u(x)=F_u(ex)=F_u(xe)\Rightarrow F_u(e)\,x=x\,F_u(e)\Rightarrow yx=xy$$

(c) $\forall x, y\in X,\ \ u, v\in U$

$$F_uF_v(xy)=F_u(F_v(x)\,y)=\big(F_uF_v(x)\big)y.$$

Also,

$$F_uF_v(xy)=F_u(x\,F_v(y))=x\big(F_uF_v(y)\big).$$

Hence F_uF_v is a translative mapping over X.

Thus, in view of (a), the system of all translative mappings over X is an abelian semigroup.

Finally, for any $x \in X$,

$$F_{u_0}(x) = F_{u_0}(ex) = F_{u_0}(e)x = ex = x.$$

Hence F_{u_0} is the identity mapping for elements of X.

Thus the proof is complete.

Corollary 1. If the system of all translative mappings over a semigroup X with identity is transitive, it is an abelian semigroup with identity where the identity is the identity mapping.

Remark. We can very easily varify, as in the above theorem, that if the system of all (rt. or lt.) translative mappings over X is transitive, it is a semigroup with identity.

Theorem 4. *Every (rt. or lt.) translative mapping over a group G is a (rt. or lt.) multiplication mapping determined by an element of G and conversely.*

PROOF. If F_u is a left translative mapping over G, then for any element $g \in G$, we have

$$F_u(g) = F_u(eg) = F_u(e)g = g_1 g$$

where e is the identity in G and $F_u(e) = g_1$.

Hence F_u is the left multiplication of G determined by g. Similarly, we can see that a rt. translative mapping over G is a rt. multiplication of G. The converse is trivial.

Hence the theorem is proved.

We note that in a group G all rt. or lt. multiplications form a transitive system of rt. or lt. translative mappings over G respectively, and hence, in an abelian group, they give rise to a transitive system of translative mappings. Further, the theorem in [1] yields the following important consequence in view of result (a) of theorem 3:

Theorem 5. *Every transitive system of translative mappings over a multiplicative semigroup X with identity has the form*

$$x \to F_u(x) = x + \Phi(u),$$

where + is an abelian group operation on X and $\Phi\colon U \to X$ *is a mapping of U onto X.*

We can easily verify that a translative mapping F_u over a semigroup X satisfies the following properties:

1. $\forall x \in X.\ F_u F_u(x^2) = F_u(x) F_u(x)$.
2. For $x, y \in X$, $F_u(x) = F_u(y)$ implies $F_u(x^i) = F_u(y^i)$ for all positive integers i.

This study is useful in connection with the study of commutable mappings over a semigroup with identity and properties of mappings in algebraic structures.

The author is much indebted to Prof. M. A. KAZIM for his valuable suggestions in the preparation of this paper.

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PUBLICATIONES MATHEMATICAE, DEBRECEN

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Felelős szerkesztő: Barna Béla
Technikai szerkesztő: Szodoray Erzsébet
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az MSZ 5601-54 és az MSZ 5602-55 szabvány szerint

A GENERALISATION TO NORMALITY

This paper gives a generalisation ina natural way to the notion of normality. We know that for any normal subset S of a group G,

(1) $\quad x S = y S \Longrightarrow S x = S y \quad$ for $x, y \in G$

The question whether a subset satisfying (1) is normal is of interest. Evidently, the answer to this question is 'no' in general. We prove in this paper that the condition (1) is equivalent to normality in case of subgroups and further give a characterization for subsets satisfying (1).

Definition 1. A subset S of a group G is called 'c-normal' if for any $x, y \in G$,

$$x S = y S \Longrightarrow S x = S y$$

Theorem 1. A subgroup H of a group G is normal if, and only if, H is c-normal.

Proof. Necessity is obvious. To prove the sufficient part, let $x \in G$ and $h \in H$ be arbitrary, then we have

$$\begin{aligned} & x H = x h H \\ \Longrightarrow\ & H x = H x h \qquad \text{(defn. 1)} \\ \Longrightarrow\ & H = H x h x^{-1} \\ \Longrightarrow\ & x H x^{-1} \subseteq H \end{aligned}$$

Similarily, if we take x^{-1} for x, we have

$$\begin{aligned} & x^{-1} H x \subseteq H \\ \Longrightarrow\ & H \subseteq x H x^{-1} \end{aligned}$$

Hence

$$H = x\,H\,x^{-1} \qquad \text{for all } x \in G$$

This completes the proof.

For giving a characterization of subsets satisfying (1), we introduce the following definition:

Definition 2. For any subset S of a group G, ' left fix group' $F_\ell(S)$ (rt. fix group $F_r(S)$) is the set of all elements $f \in G$ such that $fS = S(Sf = S)$.

In general $F_\ell(S)$ and $F_r(S)$ are distinct subgroups of G, however, if they coincide we write $F_\ell(S) = F_r(S) = F(S)$. Evidently, in case of normal subsets, left fix group and rt. fix group coincide, but, in case of c-normal subsets left fix group is contained in rt. fix group. We now prove the following theorem.

Theorem 2. A subset S of a group G is c-normal if, and only if, every subgroup conjugate to $F_\ell(S)$ is contained in $F_r(S)$.

Proof. Firstly suppose S is c-normal, then if $x \in G$ and $f \in F_\ell(S)$ be arbitrary, we have

$$\begin{aligned}
& x\,S = x\,f\,S \\
\Rightarrow\ & S\,x = S\,x\,f \quad \text{(Defn. 1)} \\
\Rightarrow\ & S = S\,x\,f\,x^{-1} \\
\Rightarrow\ & x\,f\,x^{-1} \in F_r(S) \\
\Rightarrow\ & x\,F_\ell(S)\,x^{-1} \subseteq F_r(S)
\end{aligned}$$

Conversely, if the condition of the theorem be satisfied, then let for $x, y \in G$

$$
\begin{aligned}
& xS = yS \\
\Longrightarrow\; & x^{-1}\, y\, S = S \\
\Longrightarrow\; & x^{-1}\, y \in F_{\ell}(S) \\
\Longrightarrow\; & x(x^{-1}\, y)\, x^{-1} = y\, x^{-1} \in F_r(S) \\
\Longrightarrow\; & S\, y\, x^{-1} = S \\
\Longrightarrow\; & S\, x = S\, y
\end{aligned}
$$

Hence the theorem is completely proved.

Cor.1. If S be a normal subset of a group G, then $F(S)$ is normal in G but not conversely.

Theorem 3. For any subgroup H of a group G, the following statements are equivalent:

(I) H is c-normal in G.

(II) For $x, y \in G$, $x\, H = y\, H$ implies $x\, H\, x = y\, H\, y$.

Proof. Let H satisfies (I), then

$$
\begin{aligned}
& x\, H = y\, H \Longrightarrow H\, x = H\, y \\
\Longrightarrow\; & (x\, H)(H\, x) = (y\, H)\,(H\, y) \\
\Longrightarrow\; & x\, H\, x = y\, H\, y
\end{aligned}
$$

Conversely, if H satisfies (II), then

$$
\begin{aligned}
& x\, H = y\, H \\
\Longrightarrow\; & x\, H\, x = y\, H\, y \\
\Longrightarrow\; & H\, x = x^{-1}\, y\, H\, y \\
& \quad\;\; = H\, y \qquad \text{since } x^{-1}\, y\, H = H
\end{aligned}
$$

This completes the theorem.

Note: Above theorem holds true even if H be an arbitrary subset of G.

We can easily verify that c-normality is preserved under isomorphisms in image subgroups. Our study is so far one sided. We may replace the condition (1) by

$$(2) \qquad S\,x = S\,y \implies x\,S = y\,S$$

which is again a characterization of normality of subgroups. If we call the subsets satisfying (2), c'-normal subsets of group, then it is easy to see that the theorem 2 holds for c'- normal subsets if we replace in the theorem $F_\ell(S)$ by $F_r(S)$ and vice-versa. Further theorem 3 also holds if we replace (I) and (II) by the following (I') and (II') respectively:

(I') H is c' - normal in G

(II') For $x, y \in G$, $H\,x = H\,y$ implies $x\,H\,x = y\,H\,y$.

From the duality of theorem 2, it can be easily checked that if a subset S of a group G be c-normal as well as c' - normal then $F_\ell(S) = F_r(S)$, but, even then the set may not be normal since any element not in the centre of a group is of this type.

I take this opportunity to thank Prof. M.A. Kazim for his kind help in the preparation of this paper.

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SOME IMPROVEMENTS TO MY PAPER *

The author, in [1], has defined that if X be a groupoid with respect to the operation of multiplication then a system of mappings $F_u : x \to F_u(x)$ of X into X, where u ranges over elements of a set U, is called <u>right translative</u> or <u>left translative</u> respectively according as

(1) $\forall$ x, y X, u U

$$F_u(xy) = x\, F_u(y)$$

or

$$F_u(xy) = F_u(x)y$$

or

$$F_u(xy) = F_u(x)y = x\, F_u(y)$$

holds. A system of mappings $F_u : x \to F_u(x)$ is <u>transitive</u> if

$$\forall \quad x \in X, \quad F_u(x) = X$$

holds, where the set of element $F_u(x) (u \in U)$ is denoted by $F_U(x)$. We, in this paper, give some improvements to theorems in [1] and some new results. The following theorem gives a generalisation to theorem 1 [1].

---

* The author is very thankful for the financial assistance given by C. S.I.R., Government of India, during this work.

Theorem 1. A translative mapping F_u over a quasigroup X is 1-1 if (rt. or left) cancellation law holds with respect to any element of $F_u(X)$.

Proof. We shall prove the theorem supposing that left cancellation law with respect to any element of $F_u(X)$ holds. Let left cancellation with respect to an element $F_u(x')$ in $F_u(X)$ holds, then if for $x_1, x_2 \in X$

$$\begin{aligned} & F_u(x_1) = F_u(x_2) \\ \Longrightarrow\ & x'F_u(x_1) = x'F_u(x_2) \\ \Longrightarrow\ & F_u(x')x_1 = F_u(x')x_2 \\ \Longrightarrow\ & x_1 = x_2 \end{aligned}$$

Thus F_u is 1-1. Similarly, we can prove the result in case rt. cancellation with respect to any element of $F_u(X)$ holds.

Theorem 2. If a translative mapping F_u over a quasigroup X with identity e be 1-1, then cancellation law with respect to an element $x \in X$ holds.

Proof. We put $x = F_u(e)$. Now, if for $x_1, x_2 \in X$

$$\begin{aligned} & xx_1 = xx_2 \\ \Longrightarrow\ & F_u(e)x_1 = F_u(e)x_2 \\ \Longrightarrow\ & F_u(ex_1) = F_u(ex_2) \\ & x_1 = x_2 \end{aligned}$$

Thus left cancellation with respect to x holds. Similarly, we can show that right cancellation with respect to x holds.

Hence the theorem is complete.

Theorem 3. If a translative mapping F_u over a quasigroup X be 1-1, then left (or rt.) cancellation with respect to an element $x \in X$ holds iff left (or rt.) cancellation with respect to $F_u(x)$ holds.

Proof. Obvious.

Note. In the converse part of the above theorem we do not need the 1-1 ness of F_u.

We now give a generalisation to the theorem 3(b) [1].

Theorem 4. There exists a transitive system of translative mappings $F_u (u \in U)$ over a quasigroup X with identity e if, and only if, X is an abelian group.

Proof. If X be an abelian group, then the proof is trivial. Conversely, if the condition be satisfied, then let $x, y, z \in X$ be arbitrary.

(i) Associativity : Let for $u_o \in U$

$$F_{u_o}(e) = z$$

(such a u_o always exists by supposition of transitivity), we have

$$F_{u_o}(xy) = F_{u_o}((xy)e)$$

$$\implies x F_{u_o}(y) = (xy)F_{u_o}(e)$$

$$\implies x F_{u_o}(ye) = (xy)z$$

$$\implies x(y F_{u_o}(e)) = (xy)z$$

$$\implies x(yz) = (xy)z$$

Thus the operation is associative in X.

(ii) Existence of Inverse: Let for $u' \in U$,

$$F_{u'}(x) = e \quad \text{(transitivity)}$$

$$e = F_{u'}(xe) = F_{u'}(ex)$$

$$e = x\,F_{u'}(e) = F_{u'}(e)x$$

$\Longrightarrow$ x is inversible

$\Longrightarrow$ X is inversible

Hence, from (i), (ii) and theorem 3(b) [1] it follows that X is an abelian group.

This proves the theorem.

<u>Remark.</u> From above theorem and theorem 4 [1], it follows that it is of no importance to study a transitive system of translative mappings over a quasigroup with identity.

<u>Theorem 5.</u> If $F_u (u \in U)$ be the system of all translative mappings over a quasigroup X with identity e, then $F_U(e)$ is an abelian semigroup.

<u>Proof.</u> The proof is immediate in view of theorem 3(c) [1].

<u>Theorem 6.</u> If in a quasigroup X rt. (or left) cancellation with respect to an element $x \in X$ holds then any rt. (or left) translative mapping F_u for which $F_u(x) = x$, is identity map.

<u>Proof.</u> The proof is obvious.

The author is very thankful for his discussions with Dr. K.M.Rangaswami, Mr. Amit Roy and also to Professor M.A. Kazim for his advice.

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